

THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2023-24
Homework 2
Due Date: 28th September 2023

Compulsory Part

1. When $A = \{a\}$ is a singleton, show that the free group $F(A)$ is isomorphic to the infinite cyclic group \mathbb{Z} .
2. Verify that $\mathbb{Z}^{\oplus A} := \{f : A \rightarrow \mathbb{Z} : f(a) \neq 0 \text{ for only finitely many } a \in A\}$ is indeed an abelian group, for any given set A .
3. Show that a finitely generated abelian group can be presented as a quotient of $\mathbb{Z}^{\oplus n}$ for some positive integer n .
4. Prove that $(\mathbb{Q}_{>0}, \cdot)$ is a free abelian group, meaning that it is isomorphic to $\mathbb{Z}^{\oplus A}$ for some set A .

[*Hint:* Use the fundamental theorem of arithmetic, i.e., every positive integer can be uniquely factorized as a product of primes.]

5. Let G be a group. For any $g \in G$, the map $i_g : G \rightarrow G$ defined by $i_g(a) = gag^{-1}$ for any $a \in G$ is an automorphism of G , which is called an **inner automorphism** of G . Prove that the set $\text{Inn}(G)$ of inner automorphisms of G is a normal subgroup of the automorphism group $\text{Aut}(G)$ of G .

[*Warning:* Be sure to show that the inner automorphisms do form a subgroup.]

6. Show that an intersection of normal subgroups of a group G is again a normal subgroup of G .
7. Let G be a group containing at least one subgroup of a fixed finite order s . Show that the intersection of all subgroups of G of order s is a normal subgroup of G .

[*Hint:* Use the fact that if H has order s , then so does $x^{-1}Hx$ for all $x \in G$.]

Optional Part

1. Let G be a finite group with $|G|$ odd. Show that the equation $x^2 = a$, where x is the indeterminate and a is any element in G , always has a solution. (In other words, every element in G is a square.)
2. Generalizing the above question: If G is a finite group of order n and k is an integer relatively prime to n , show that the map $G \rightarrow G, a \mapsto a^k$ is surjective.
3. Prove that every finite group is finitely presented.
4. We have learnt that a presentation of the dihedral group D_n is given by $(a, b \mid a^2, b^n)$
5. Let a, b be distinct elements of order 2 in a group G . Suppose that ab has finite order $n \geq 3$. Prove that the subgroup $\langle a, b \rangle$ generated by a and b is isomorphic to the dihedral group D_n (which has $2n$ elements).
6. Let $G = \mathbb{Z}^{\oplus \mathbb{N}}$. Prove that $G \times G \cong G$ (as abelian groups).
7. Prove that $(\mathbb{Q}, +)$ is not a free abelian group.
8. Show that if a finite group G has exactly one subgroup H of a given order, then H is a normal subgroup of G .
9. Show that the set of all $g \in G$ such that the inner automorphism $i_g : G \rightarrow G$ is the identity inner automorphism i_e is a normal subgroup of a group G .