Hw2 Math 2230 B/C

P61. 4. $f'(z_0) = \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$

$\therefore \dfrac{f'(z_0)}{g'(z_0)} = \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} \cdot \dfrac{z - z_0}{g(z) - g(z_0)} = \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{g(z) - g(z_0)} = \lim_{z \to z_0} \dfrac{f(z)}{g(z)}$

9. $f(dx, 0) = \dfrac{dx^2}{dx} = dx \qquad \dfrac{f(dx, 0) - f(0, 0)}{dx - 0} = \dfrac{dx}{dx} = 1$ (horizontal)

$f(0, dy) = \dfrac{(-idy)^2}{idy} = idy \qquad \dfrac{f(0, dy) - f(0, 0)}{idy - 0} = \dfrac{idy}{idy} = 1$ (vertical)

$f(dx, dx) = \dfrac{(dx - idx)^2}{dx + idx} = \dfrac{(1-i)^2}{(1+i)} dx$

$\dfrac{f(dx, dx) - f(0, 0)}{dx + idx - 0} = \dfrac{(1-i)^2}{(1+i)^2} = \dfrac{-2i}{2i} = -1$

P70. 1. let $z = x + iy$

a) $f(z) = \bar{z} = x - iy \qquad u(x, y) = x \qquad v(x, y) = -y$

$\Rightarrow u_x = 1 \quad u_y = 0 \quad v_x = 0 \quad v_y = -1$

$\Rightarrow u_x \neq v_y \quad \forall x, y \in \mathbb{R}. \Rightarrow f'(z)$ DNE

b) $f(z) = z - \bar{z} = i2y, \quad u(x, y) = 0. \quad v(x, y) = 2y$

$\Rightarrow u_x = 0, \quad u_y = 0 \quad v_x = 0. \quad v_y = 2$

$\Rightarrow u_x \neq v_y \quad \forall x, y \in \mathbb{R} \Rightarrow f'(z)$ DNE

c) $f(z) = 2x + ixy^2 \qquad u(x, y) = 2x \qquad v(x, y) = xy^2$

$\Rightarrow u_x = 2 \quad u_y = 0 \quad v_x = y^2$ 且 $v_y = 2xy$

If $u_x = v_y$, $u_y = -v_x$, Then $\begin{cases} 2xy = 2 \\ -y^2 = 0 \end{cases}$ such x, y DNE

$\therefore f'(z)$ DNE

d) $f(z) = e^x e^{-iy} = e^x \cos y - i e^x \sin y$

$u_x = e^x \cos y$, $u_y = -e^x \sin y$, $v_x = -e^x \sin y$, $v_y = -e^x \cos y$

If $u_x = v_y$, $u_y = -v_x$, then

$$\begin{cases} e^x \cos y = -e^x \cos y & \Rightarrow y = \frac{\pi}{2} + k\pi \\ e^x \sin y = -e^x \sin y & \Rightarrow y = k\pi \end{cases}$$

such y DNE

$\Rightarrow f'(z)$ DNE

3. let $z = x+iy$

a) $f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

$u(x,y) = \frac{x}{x^2+y^2}$ $\quad v(x,y) = \frac{-y}{x^2+y^2}$

$u_x = \frac{y^2-x^2}{(x^2+y^2)^2}$, $u_y = \frac{-2xy}{(x^2+y^2)^2}$, $v_x = \frac{2xy}{(x^2+y^2)^2}$, $v_y = \frac{y^2-x^2}{(x^2+y^2)^2}$

$$f'(z) = u_x + iv_x = \frac{y^2-x^2+i2xy}{(x^2+y^2)^2} = \frac{-(\bar{z})^2}{(z\bar{z})^2} = -\frac{1}{z^2} \quad (z \neq 0)$$

b) $f(z) = x^2 + iy^2$, $u(x,y) = x^2$, $v(x,y) = y^2$

$u_x = 2x$, $u_y = 0$, $v_x = 0$, $v_y = 2y$

let $u_x = v_y$, $2x = 2y \Rightarrow x = y$

$v_y = -v_x = 0$

$\Rightarrow f'(z) = u_x + iv_x = 2x$ on $\{(x,y) \in \mathbb{R}^2 \mid x = y\}$

c) $f(z) = z\,\mathrm{Im}\,z = (x+iy)y = xy + iy^2$, $u(x,y) = xy$, $v(x,y) = y^2$

$u_x = y$, $u_y = x$, $v_x = 0$, $v_y = 2y$

let $u_x = v_y \Rightarrow y = 0$

$u_y = -v_x \Rightarrow x = 0$

$\Rightarrow f'(z) = u_x + iv_x = 0$ on $z = 0$

P76.7. $f(z) = u(x,y) + iv(x,y) = u(x,y)$ since f is real-valued

Given f is analytic, by C-R eq'ns

$$u_x = v_y = 0 \qquad u_y = -v_x = 0$$

$\Rightarrow u(x,y) = C$ is a constant function

$\Rightarrow f(z)$ is constant

P89.3. let $z = x+iy$

$f(z) = \exp \bar{z} = e^{x-iy} = e^x \cos y - ie^x \sin y$

$u(x,y) = e^x \cos y$, $v(x,y) = -e^x \sin y$

$u_x = e^x \cos y \quad u_y = -e^x \sin y \quad v_x = -e^x \sin y \quad v_y = -e^x \cos y$

If $u_x = v_y$, $u_y = -v_x$, then $e^x \cos y = -e^x \cos y \Rightarrow y = \frac{\pi}{2} + k\pi$

such y DNE

$-e^x \sin y = e^x \sin y \Rightarrow y = k\pi$

$\Rightarrow f'(z)$ DNE

4. Method 1: $f(z) = z^2 \quad g(z) = e^z$ then $f(z) = g \circ f(z)$ is entire.

Method 2: $u(x,y) = e^{x^2-y^2} \cos 2xy \qquad v(x,y) = e^{x^2-y^2} \sin 2xy$

$u_x = 2xe^{x^2-y^2} \cos 2xy - 2ye^{x^2-y^2} \sin 2xy$

$u_y = -2ye^{x^2-y^2} \cos 2xy - 2xe^{x^2-y^2} \sin 2xy$

$v_x = 2xe^{x^2-y^2} \cos 2xy + 2ye^{x^2-y^2} \cos 2xy$

$v_y = -2ye^{x^2-y^2} \sin 2xy + 2xe^{x^2-y^2} \cos 2xy$

so f is entire $\qquad f'(z) = 2ze^{z^2}$

P96.6. $e^{\log z} = z \quad (|z|>0,\ \alpha < \arg z < \alpha + 2\pi)$

$\Rightarrow \frac{d}{dz} e^{\log z} = \frac{d}{dz} z$

$\Rightarrow \left(\frac{d}{dz}(\log z)\right) \cdot e^{\log z} = 1$

$\Rightarrow \frac{d}{dz}\log z = \left(e^{\log z}\right)^{-1} = \frac{1}{z}$

10. a) $z - i \neq 0 \Rightarrow z \neq i$

since $\log z$ is undefined at $z=0$

let $z = x+iy$

$\mathrm{Log}(z)$ is analytic for $|z|>0$, $-\pi < \mathrm{Arg} z < \pi$

$\Rightarrow \mathrm{Arg}(z-i) = \mathrm{Arg}(x+i(y-1)) = \pi$ (brunch cut)

$\Rightarrow x<0$ and $y=1$ combine with $z \neq i$, then $f(z)$ is analytic except on $\{x \leq 0, y=1\}$.

b) $z^2 + i = 0 \Rightarrow (x+iy)^2 + i = 0$

$\Rightarrow x^2 - y^2 + i(1+2xy) = 0$

$\Rightarrow x^2 = y^2, \quad 2xy + 1 = 0$

$\Rightarrow x+iy = \frac{1-i}{\sqrt{2}}$ or $x+iy = \frac{-1+i}{\sqrt{2}}$

$z+4 = 0 \Rightarrow z = -4$

$\mathrm{Arg}(z+4) = \mathrm{Arg}(x+4+iy) = \pi$ (brunch cut)

$\Rightarrow x+4 < 0, \quad y = 0$

$\Rightarrow f(z)$ analytic except at $\pm\frac{1-i}{\sqrt{2}}$ and $\{x \leq -4, y=0\}$

P103. 1. a) $(1+i)^i = e^{i\log(1+i)} = e^{i(\ln\sqrt{2} + (\frac{\pi}{4}+2k\pi)i)}$

$$\overset{n=-k}{=} \exp(-\frac{\pi}{4} + 2n\pi)\exp(i\frac{\ln 2}{2})$$

b) $\frac{1}{i^{2i}} = i^{-2i} = e^{-2i\log i} = e^{-2i((\frac{\pi}{2}+2n\pi)i)}$, $n=0, \pm1, \pm2$

$$\overset{n=-k}{=} \exp(\pi + 4n\pi)$$

$$= \exp[(4n+1)\pi]$$

6. P.V. $z^a = e^{a\log z} = e^{a(\ln|z| + i\operatorname{Arg}z)}$, $|z|>0$, $-\pi < \operatorname{Arg}z \le \pi$

$$\Rightarrow |z^a| = |e^{a\ln|z|} \cdot e^{ia\operatorname{Arg}z}|$$

$$= |e^{a\ln|z|}| \cdot |e^{ia\operatorname{Arg}z}|$$

$$= |z|^a$$

since $|e^{i\theta}| = 1$, $\forall \theta \in \mathbb{R}$

9. $c^{f(z)} = \exp(f(z)\log c)$

$$\Rightarrow \frac{d}{dz} c^{f(z)} = \exp(f(z)\log c) \cdot \log c \cdot f'(z)$$

$$= c^{f(z)} \cdot \log c \cdot f'(z)$$