Hw6 Math 2230 B/C

P237.3 $f: \frac{4z-5}{z(z-1)} \Rightarrow \frac{1}{z^2} f(\frac{1}{z}) = \frac{1}{z^2} \cdot \frac{\frac{4}{z}-5}{\frac{1}{z^2}-\frac{1}{z}} = \frac{\frac{4}{z}-5}{1-z} = \frac{\frac{4-5z}{1-z}}{z}$

$\underset{z=0}{\text{Res}} \left(\frac{1}{z^2} f(\frac{1}{z})\right) = 4 \quad \therefore \int_c f\,dz = 2\pi i \cdot 4 = 8\pi i$

P237.4 a) $f(z) = \frac{z^5}{1-z^2} \Rightarrow \frac{1}{z^2} f(\frac{1}{z}) = \frac{z^{-5}}{1-z^{-2}} \cdot \frac{1}{z^2} = \frac{1}{z^2-1} \cdot \frac{1}{z^5}$

$= -\sum_{n=0}^{\infty} z^{2n} \cdot z^{-5}$

$= -z^{-5} - z^{-3} - z^{-1} - \cdots$

$\therefore \underset{z=0}{\text{Res}} \left(\frac{1}{z^2} f(\frac{1}{z})\right) = -1 \qquad \int_c f\,dz = -2\pi i$

b) $f(z) = \frac{1}{1+z^2} \Rightarrow \frac{1}{z^2} \cdot \frac{1}{1+z^{-2}} = \frac{1}{z^2+1} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$

$\therefore \int_c f\,dz = 0$

c) $f(z) = \frac{1}{z} \Rightarrow z^{-2} f(\frac{1}{z}) = \frac{1}{z} \Rightarrow \int_c f(z)\,dz = 2\pi i$

P237.7. $\int_C \frac{P(z)}{Q(z)}\,dz = 2\pi i \,\text{Res}\left(\frac{1}{z^2} \frac{P(\frac{1}{z})}{Q(\frac{1}{z})}, 0\right)$

$\frac{1}{z^2} \frac{P(\frac{1}{z})}{Q(\frac{1}{z})} = \frac{a_0 + a_1 \frac{1}{z} + \cdots + a_n z^{-n}}{z^2(b_0 + b_1 \frac{1}{z} + \cdots + b_m z^{-m})} \cdot \frac{z^{m-2}}{z^{m-2}}$

$= \frac{a_0 z^{m-2} + a_1 z^{m-3} + \cdots + a_n z^{m-n-2}}{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}$, $m-n-2 \geq 0 \quad z \neq 0$

since $b_0 z^m + b_1 z^{m-1} + \cdots + b_m = b_m$ when $z \to 0$

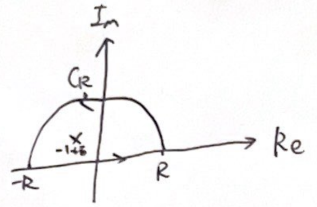
$\therefore b_0 z^m + b_1 z^{m-1} + \cdots + b_m \neq 0$ in $B_\varepsilon(0)$ for some $\varepsilon > 0$

$\therefore \frac{1}{z^2} \frac{P(\frac{1}{z})}{Q(\frac{1}{z})}$ is analytic in $B_\varepsilon(0)$

$\Rightarrow \int_C \frac{P(z)}{Q(z)}\,dz = 0$

P265 · 7. $\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{1}{x^2+2x+2}\,dx$

we use the simple closed curve



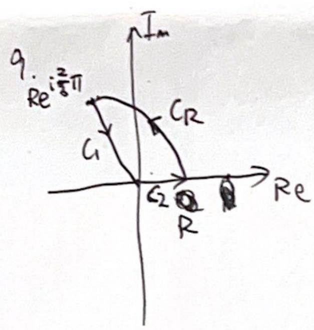
$$\int_{-R}^{R}\frac{1}{x^2+2x+2}\,dx + \int_{C_R}\frac{z}{z^2+2z+2}\,dz = 2\pi i\ Res(f,-1+i)$$

$$Res(f,-1+i) = \frac{1}{2i}$$

and $\left|\int_{C_R}\frac{1}{z^2+2z+2}\,dz\right| \le 2\pi R\,\frac{2}{R^2} \to 0$ as $R\to\infty$

$$\therefore \int_{-\infty}^{\infty}\frac{1}{x^2+2x+2}\,dx = 2\pi i\cdot\frac{1}{2i} = \pi$$

9.



$$\int_{C_1}+\int_{C_2}+\int_{C_R}\frac{1}{x^3+1}\,dx = 2\pi i\ Res(f, e^{i\frac{\pi}{3}})$$

and $\left|\int_{C_R}\frac{1}{z^3+1}\,dz\right| \le \frac{2}{3}\pi R\cdot\frac{2}{R^3} \to 0$ as $R\to\infty$

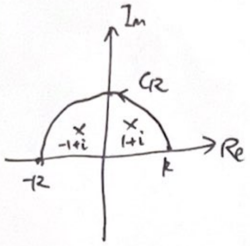
$$\int_{C_1}\frac{1}{z^3+1}\,dz = \int_R^0\frac{e^{i\frac{2}{3}\pi}}{(re^{i\frac{2}{3}\pi})^3+1}\,dr = -e^{i\frac{2}{3}\pi}\int_0^R\frac{1}{r^3+1}\,dr$$

$$= e^{-\frac{2}{3}\pi}\int_0^R\frac{1}{x^3+1}\,dx$$

$$Res(f, e^{i\frac{\pi}{2}}) = \frac{1}{3e^{i\frac{2\pi}{3}}}$$

$$\therefore (1-e^{i\frac{2}{3}\pi})\int_0^R\frac{1}{x^3+1}\,dx = 2\pi i\cdot\frac{1}{3}\cdot\frac{1}{-\frac{1}{2}+\frac{\sqrt{3}}{2}i} = \frac{2\pi}{3\sqrt{3}}$$

P273.5. we use the simple closed curve:



we have $\int_{-R}^{R} \frac{e^{iax}x^3}{x^4+4}dx + \int_{C_R} \frac{e^{iaz}z^3}{z^4+4}dz$

$$= 2\pi i\left[\operatorname{Res}(f, 1+i) + \operatorname{Res}(f, -1+i)\right]$$

$\lim_{R\to\infty} \left|\int_{C_R} \frac{e^{-iaz}z^3}{z^4+4}dz\right| \cancel{\leq \frac{2R^3}{R^4}} = 0$ by Jordan's lemma

and $\int_{-R}^{R} \frac{e^{iax}x^3}{x^4+4}dx = \int_{-R}^{R} \frac{\cos ax \cdot x^3}{x^4+4}dx + i\int_{-R}^{R} \frac{\sin ax \cdot x^3}{x^4+4}dx$

$\operatorname{Res}(f, 1+i) = \frac{(1+i)^3 e^{ia(1+i)}}{4(1+i)^3} = \frac{e^{-a}e^{ia}}{4}$

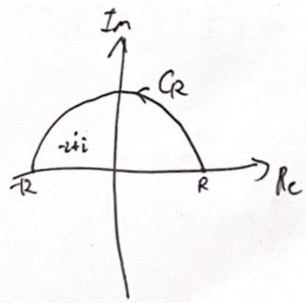
$\operatorname{Res}(f, i-1) = \frac{(i-1)^3 e^{ia(i-1)}}{4(i-1)^3} = \frac{e^{-a}e^{-ia}}{4}$

$\therefore i\int_{-R}^{R} \frac{\sin ax \cdot x^3}{x^4+4}dx = 2\pi i\left(\frac{e^{-a}e^{ia}}{4} + \frac{e^{-a}e^{-ia}}{4}\right)$

$\Rightarrow \int_{-R}^{R} \frac{\sin ax \cdot x^3}{x^4+4}dx = \pi e^{-a}\cos a$

$$\int_{-\infty}^{\infty} \frac{\sin ax \cdot x^3}{x^4+4}dx = \pi e^{-a}\cos a$$

10. we use the simple closed curve:



we have $\int_{-R}^{R} \frac{e^{ix}(x+1)}{x^2+4x+5}dx + \int_{C_R} \frac{e^{iz}(z+1)}{z^2+4z+5}dz$
$= 2\pi i \operatorname{Res}(f, -2+i)$

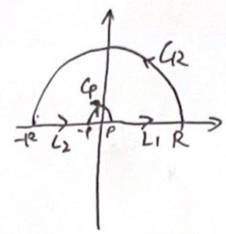
$\lim_{R\to\infty} \left|\int_{C_R} \frac{e^{iz}(z+1)}{z^2+4z+5}dz\right| = 0$ by Jordan's lemma

$\operatorname{Res}(f, -2+i) = \frac{e^{i(i-2)}(i-1)}{2(-2+i)+4} = \frac{e^{-2i-1}(i-1)}{2i}$

$\therefore \int_{-R}^{R} \frac{\cos x(x+1)}{x^2+4x+5}dx + i\int_{-R}^{R} \frac{\sin x(x+1)}{x^2+4x+5}dx = \frac{\pi}{e}(\sin 2 - \cos 2) + i\frac{\pi}{e}(\sin 2 + \cos 2)$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos x(x+1)}{x^2+4x+5}dx = \frac{\pi}{e}(\sin 2 - \cos 2)$$

P282.1.



Apply Cauchy's theorem to the function $f(z) = \frac{e^{iaz} - e^{ibz}}{z^2}$

we have $\int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz = 0$

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = \int_\rho^R \frac{e^{iar} - e^{ibr}}{r^2}\,dr + \int_\rho^R \frac{e^{-iar} - e^{-ibr}}{r^2}\,dr = 2\int_\rho^R \frac{\cos(ar) - \cos(br)}{r^2}\,dr$$

$\left|\int_{C_R} f(z)\,dz\right| \le \frac{2}{R^2}\pi R = \frac{2\pi}{R} \to 0$ as $R \to \infty$

and $f(z) = \frac{1}{z^2}\left[\left(1 + \frac{iaz}{1} + \frac{(iaz)^2}{2!} + \cdots\right) - \left(1 + \frac{ibz}{1} + \frac{(ibz)^2}{2!} + \cdots\right)\right]$

$= \frac{i(a-b)}{z} + O(z^2)$

$\therefore \lim_{\rho \to 0} \int_{C_\rho} f(z)\,dz = i(a-b)\int_{C_\rho} \frac{1}{z}\,dz = -\pi i\,[i(a-b)] = \pi(a-b)$

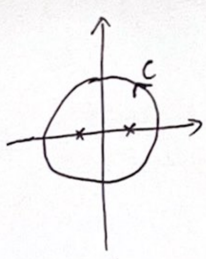
$\Rightarrow \lim_{\rho \to 0}\lim_{R \to \infty} 2\int_\rho^R \frac{\cos ar - \cos br}{r^2}\,dr + \pi(a-b) = 0$

$\Rightarrow \int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2}\,dx = \frac{\pi}{2}(b-a)$

Take $a=0$ $b=2$ we have $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{1}{2}\int_0^\infty \frac{1 - \cos 2x}{x^2}\,dx = \frac{\pi}{2}$

P287.2. $\sin^2\theta = \left(\frac{z - z^{-1}}{2i}\right)^2$

$\therefore \int_{-\pi}^{\pi} \frac{1}{1 + \sin^2\theta}\,d\theta = \int_C \frac{1}{1 + \left(\frac{z - z^{-1}}{2i}\right)^2} \cdot \frac{1}{iz}\,dz = \int_C \frac{4iz}{z^4 - 6z^2 + 1}\,dz$



$\mathrm{Res}(f, \sqrt{3 - 2\sqrt{2}}) = \frac{4i(\sqrt{3-2\sqrt{2}})}{4(\sqrt{3-2\sqrt{2}})^3 - 12\sqrt{3-2\sqrt{2}}} = -\frac{i}{2\sqrt{2}}$

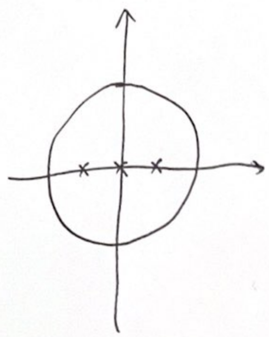
$\mathrm{Res}(f, -\sqrt{3 - 2\sqrt{2}}) = \frac{4i \cdot (-\sqrt{3-2\sqrt{2}})}{4(-\sqrt{3-2\sqrt{2}})^3 - 12(-\sqrt{3-2\sqrt{2}})} = -\frac{i}{2\sqrt{2}}$

$\therefore \int_{-\pi}^{\pi} \frac{1}{1 + \sin^2\theta}\,d\theta = 2\pi i \cdot -\frac{i}{\sqrt{2}} = \sqrt{2}\pi$

P287.3. $\cos 3\theta = \frac{e^{i3\theta}+e^{i-3\theta}}{2}$ $\quad \cos 2\theta = \frac{e^{i2\theta}+e^{-i2\theta}}{2}$

let $z=e^{i\theta}$ $\quad dz = iz\,d\theta$ $\quad \cos^2(3\theta) = \left(\frac{z^3+z^{-3}}{2}\right)^2$

$$\int_0^{2\pi}\frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta = \frac{1}{4}\int_C \frac{z^6+2+z^{-6}}{5-4\left(\frac{z^2+z^{-2}}{2}\right)}\cdot\frac{1}{iz}\,dz = \int_{|z|=1}\frac{z^{12}+2z^6+1}{20z^6-8z^8-8z^4}\,\frac{dz}{iz}$$



There are 3 poles at $-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}$

let $f = \frac{z^{12}+2z^6+1}{z^5(5z^2-2z^4-2)} = \frac{\phi(z)}{z^5}$ where $\phi(z) = \frac{z^{12}+2z^6+1}{5z^2-2z^4-2}$

$\therefore \phi(z) = -\frac{1}{2}z^8 - \frac{5}{4}z^6 - \frac{21}{8}z^4 - \cdots$

$\therefore \operatorname{Res}(f,0) = \frac{\phi^{(4)}(0)}{4!} = -\frac{21}{8}$

$\operatorname{Res}\left(f,\frac{\sqrt{2}}{2}\right) = \left.\frac{z^{12}+2z^6+1}{35z^6-18z^8-10z^4}\right|_{z=\frac{\sqrt{2}}{2}} = \frac{27}{16}$

$\operatorname{Res}\left(f,-\frac{\sqrt{2}}{2}\right) = \left.\frac{z^{12}+2z^6+1}{35z^6-18z^8-10z^4}\right|_{z=-\frac{\sqrt{2}}{2}} = \frac{27}{16}$

$$\therefore \int_0^{2\pi}\frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta = \frac{1}{4i}\int_{|z|=1} f(z)\,dz = 2\pi i\cdot\frac{1}{4i}\left[-\frac{21}{8}+\frac{27}{8}\right] = \frac{3}{8}\pi$$

P293.5 near each zero z_k $\quad f(z) = (z-z_0)^{m_k}g(z)$ where $g(z)$ is analytic and non-zero

$$\therefore \frac{zf'(z)}{f(z)} = \frac{m_k z}{z-z_k} + \frac{zg'(z)}{g(z)} = \underbrace{\boxed{m_k + \frac{zg'(z)}{g(z)}}}_{\text{Analytic}} + \frac{m_k z_k}{z-z_k}$$

$$\therefore \int_C \frac{zf'(z)}{f(z)}\,dz = 2\pi i\sum_{k=1}^{n} m_k z_k$$

8. let $f_1(z) = 2z^5$ $\quad g_1(z) = -6z^2+z+1$; on $|z|=2$ $\quad |f(z)| = 2\cdot 2^5 = 64$ $\quad |g(z)| \le 6\cdot 2^2+2+1 = 27$

$\therefore f_1(z)+g_1(z) = 2z^5-6z^2+z+1$ has the same numbers of zeros as $f_1(z)$ inside $|z|=2$ which is 5.

let $f_2(z) = -6z^2$ $\quad g_2(z) = 2z^5+z+1$; on $|z|=1$ $\quad |f_2(z)| = 6$ $\quad |g_2(z)| \le 4$

$\therefore f_2(z)+g_2(z) = 2z^5-6z^2+1+z$ has 2 zeros inside $|z|=1$.

$\therefore 2z^5-6z^2+z+1$ has 3 zeros in $\{z : 1<|z|<2\}$.