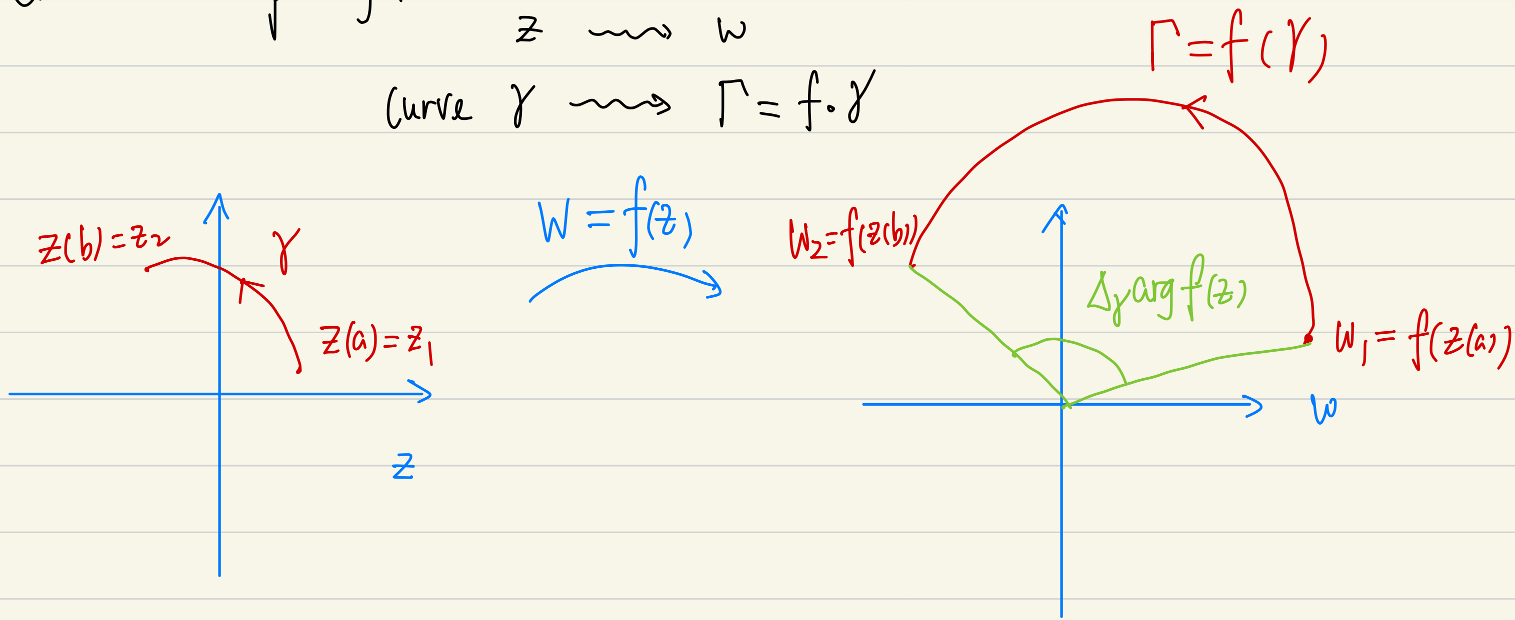
Note 8

§ Application II: Argument Principle

Given a map $f: \mathbb{C} \longrightarrow \mathbb{C}$

$z \rightsquigarrow w$

curve $\gamma \rightsquigarrow \Gamma = f \circ \gamma$



Ex: (1) $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$, the unit circle.

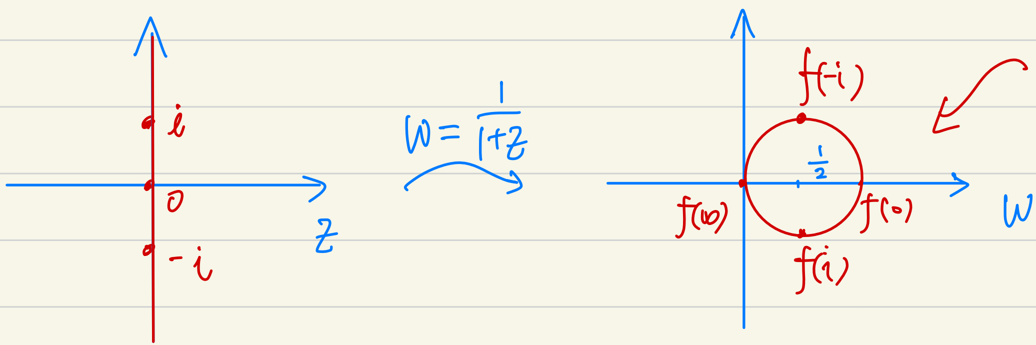
Let $f(z) = z^2$. Then $f \circ \gamma(t) = e^{i \cdot 2t}$ traverse the unit circle twice.

(2) $\gamma(t) = it$, $-\infty < t < \infty$, the y-axis.

Let $f(z) = \frac{1}{z+1}$. $\quad f \circ \gamma(t) = \frac{1}{it+1} = \frac{1-it}{1+t^2}$

Note that $f \circ \gamma(-1) = f(-i) = \frac{1+i}{2}$ $\qquad f \circ \gamma(1) = f(i) = \frac{1-i}{2}$

$f \circ \gamma(0) = f(0) = 1$ $\qquad f \circ \gamma(\infty) = f(\infty i) = 0$



$\Gamma = f(\gamma)$ is a circle of radius $\frac{1}{2}$

Now, Suppose $w = f(z)$ is holomorphic and has no zero on γ

i.e., $f(\gamma(t)) = \Gamma(t) \neq 0 \quad \forall t$

Let $\Delta_\gamma \arg f(z)$ be the continuous change in $\arg f(z)$ along γ from z_1 to z_2

that is, $\boxed{\Delta_\gamma \arg f(z) = \theta(b) - \theta(a)}$

where $f(z(t)) = \rho(t) \cdot e^{i\theta(t)}$. $t \in [a,b]$

and ρ, θ are continuous.

Ex. (1) $\Delta_\gamma \arg f(z) = 4\pi$

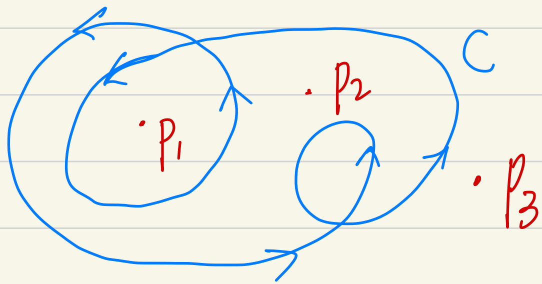
(2) Suppose $\gamma(t) = it$, $t \in [-1,1]$. $\Delta_\gamma \arg f(z) = -\frac{\pi}{2}$.

§ Winding number / index :

Let C be a closed curve , p a point NOT on C .

Define: $\mathrm{Ind}(C, p) :=$ total # times C winds counter-clockwise around p
(Geometric)

Ex. $\mathrm{Ind}(C, p_1) = 2$, $\mathrm{Ind}(C, p_2) = 1$, $\mathrm{Ind}(C, p_3) = 0$.



Algebraic Definition :

Theorem : $\mathrm{Ind}(C, z_0) = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{1}{z - z_0}\, dz$ $(*)$

This is essentially the Cauchy Integral formula.

Back to the map $w = f(z)$, $\gamma = C$, $\Gamma = f(\gamma)$ closed curve

Winding number $\mathrm{Ind}(\Gamma, w_0) \overset{\text{by def.}}{=} \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \frac{1}{w - w_0}\, dw$

Change of variable $w = f(z)$, then $dw = f'(z)\, dz$.

$$\Rightarrow \boxed{\mathrm{Ind}(\Gamma, w_0) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z) - w_0}\, dz}$$

In particular, when $w_0 = 0$, we have:

$$\boxed{\frac{1}{2\pi} \cdot \Delta_\gamma \arg f(z) = \mathrm{Ind}(f \circ \gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz}$$

Def: A function f is **meromorphic** on D if f is holomorphic on D, except for poles.

Theorem (Argument Principle): Suppose f is meromorphic on D and has no pole or zero on a simple closed curve C.

Then,

$$\mathrm{Ind}(f \circ C, 0) = \frac{1}{2\pi} \Delta_C \arg f(z) = \boxed{\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = Z - P}$$

where Z = # of zeroes of f inside C

P = # .. poles - - - - - - -, both counting multiplicities.

Pf: We determine the singularities and residues of $\dfrac{f'(z)}{f(z)}$.

- Suppose $f(z)$ has a zero z_0 of order m, then $f(z) = (z-z_0)^m \cdot g(z)$.

holomorphic, nonzero near z_0

$$\frac{f'(z)}{f(z)} = \frac{m\cdot(z-z_0)^{m-1}\cdot g(z) + (z-z_0)^m \cdot g'(z)}{(z-z_0)^m \cdot g(z)}$$

$$= \frac{m}{z-z_0} + \frac{g'(z)}{g(z)}$$

holomorphic near z_0 since $g(z) \neq 0$

Thus, z_0 is a simple pole of $\dfrac{f'(z)}{f(z)}$

and $\boxed{\operatorname{Res}\left(\frac{f'(z)}{f(z)}, z_0\right) = m = \operatorname{mult}(z_0)}$

- Similarly, if $f(z)$ has a pole P_0 of order n. then $f(z) = (z-P_0)^{-n} \cdot g(z)$

holomorphic nonzero near P_0

$$\frac{f'(z)}{f(z)} = \frac{-n(z-P_0)^{-n-1} \cdot g(z) + (z-P_0)^{-n} \cdot g'(z)}{(z-P_0)^{-n} \cdot g(z)}$$

$$= \frac{-n}{z-P_0} + \frac{g'(z)}{g(z)}$$

holomorphic near P_0

Thus, P_0 is a simple pole of $\frac{f'(z)}{f(z)}$

and $\boxed{\mathrm{Res}\left(\frac{f'(z)}{f(z)}, P_0\right) = -n = -\mathrm{mult}(P_0)}$

- If z is neither a zero nor a pole of $f(z)$,

  then $\dfrac{f'(z)}{f(z)}$ is holomorphic at z.

Finally, apply the Residue Theorem:

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{\substack{\text{sing. of } \frac{f'}{f} \\ \text{inside } C}} \operatorname{Res}\left(\frac{f'}{f}\right)$$

$$= \sum_{\substack{\text{zero of } f \\ \text{inside } C}} \operatorname{mult}(z_k) - \sum_{\substack{\text{pole of } f \\ \text{inside } C}} \operatorname{mult}(p_k)$$

$$= Z - P$$

$\square$

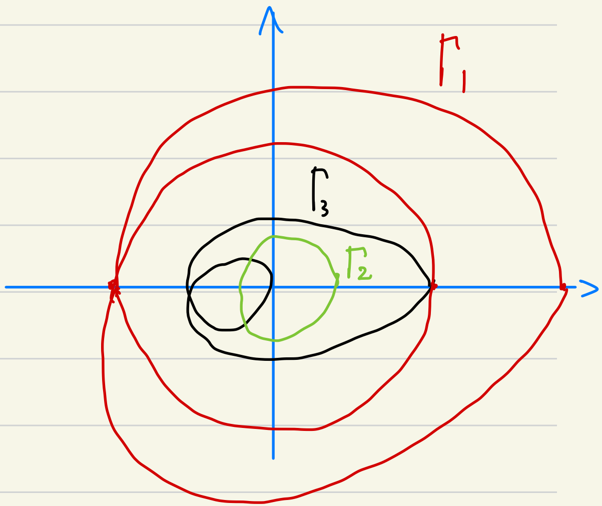
Ex: • $f(z) = z^2$ $\quad C$ = unit circle

C enclose $z_0 = 0$ a zero of order 2.

$\Rightarrow$ $\mathrm{Ind}(f \circ C, 0) = Z - P = 2$.

$\square$

• $f(z) = z^2 + z = z(z+1)$



(i) C_1 = Circle of radius $\frac{1}{2}$. $\mathrm{Ind}(f \circ C_1, 0) = 1 - 0 = 1$

(ii) C_2 = circle of radius 2. $\mathrm{Ind}(f \circ C_2, 0) = 2 - 0 = 2$

(iii) C_3 = unit circle. the zero $z_0 = -1$ lies on C_3

so argument principle doesn't apply!

§ Rouché's Theorem.

Theorem Let C be a simple closed curve. Suppose

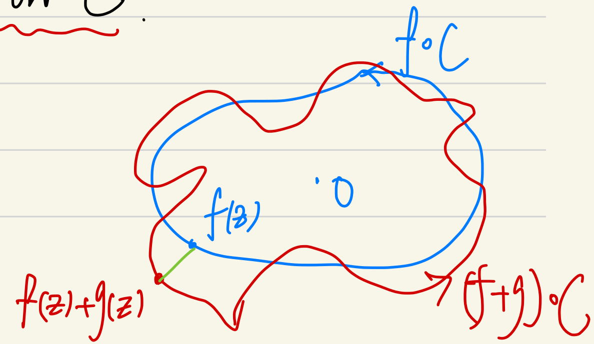
(a) $f(z)$ and $g(z)$ are holomorphic inside and on C.

(b) $|f(z)| > |g(z)|$ for $z \in C$.

Then $f(z)$ and $f(z)+g(z)$ have the same number of zeroes inside C counting multiplicities.

More generally, Suppose f and g are meromorphic & no pole on C.

Then
$$\underset{\displaystyle \mathrm{Ind}(f\circ C,\ 0)}{\underset{\|}{Z_f - P_f}} = \underset{\displaystyle \mathrm{Ind}((f+g)\circ C,\ 0)}{\underset{\|}{Z_{f+g} - P_{f+g}}}.$$



Pf: Note that $|f(z)| > |g(z)| \geq 0$ $\forall z \in C$

$$|f(z)+g(z)| \geq |f(z)| - |g(z)| > 0.$$

$\Rightarrow$ Neither $f(z)$ nor $f(z)+g(z)$ has zero on C

By assumption, 〃 〃 pole on C.

Thus, we can apply the argument principle on f and $f+g$.

$$\operatorname{Ind}(f\circ C, 0) = \frac{1}{2\pi}\Delta_C \arg f(z) = \frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = Z_f - P_f \qquad (1)$$

$$\operatorname{Ind}((f+g)\circ C, 0) = \frac{1}{2\pi}\Delta_C \arg(f(z)+g(z)) = \frac{1}{2\pi i}\int_C \frac{(f+g)'}{f+g}\,dz = Z_{f+g} - P_{f+g}. \qquad (2)$$

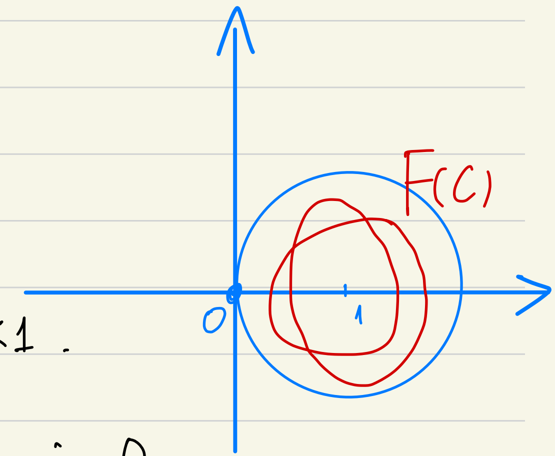
- Compare $\mathrm{Ind}((f+g)\circ C, 0)$ and $\mathrm{Ind}(f\circ C, 0)$

Note that $f+g = f\cdot\underbrace{\left(1+\frac{g}{f}\right)}_{= F} = f\cdot F$

Computation shows $\frac{(f+g)'}{f+g} = \frac{(f\cdot F)'}{f\cdot F} = \frac{f'}{f} + \frac{F'}{F}$

$$\Rightarrow \boxed{\mathrm{Ind}((f+g)\circ C, 0) = \mathrm{Ind}(f\circ C, 0) + \mathrm{Ind}(F\circ C, 0)}$$

- Prove: $\operatorname{Ind}(\overset{= 1+\frac{g}{f}}{F} \circ C, 0) = 0$



Note $|F(z) - 1| = \left|\frac{g(z)}{f(z)}\right| < 1 \quad \forall z \in C$

$\Rightarrow$ F maps the curve C inside the disk $|w-1|<1$.

From geometry, it is clear that winding number is 0.

$\square$

$P(z)$ $\parallel$

Ex: Find the distribution of roots of $z^4+3z^3+6=0$ (counting multiplicities)

Sol: • Let $C_1: |z|=10$. $f(z)=z^4$, $g(z)=3z^3+6$

On $|z|=10$. $|g(z)| \le 3006 < 10000 = |f(z)|$

As $f(z)$ has a root of multiplicity 4 inside C_1

Rouche's thm $\Rightarrow$ 4 roots of $f+g$ inside C_1.

• Let $C_2: |z|=2$. $f(z)=3z^3$. $g(z)=z^4+6$

On $|z|=2$, $|g(z)| \le 22 < 24 = |f(z)|$.

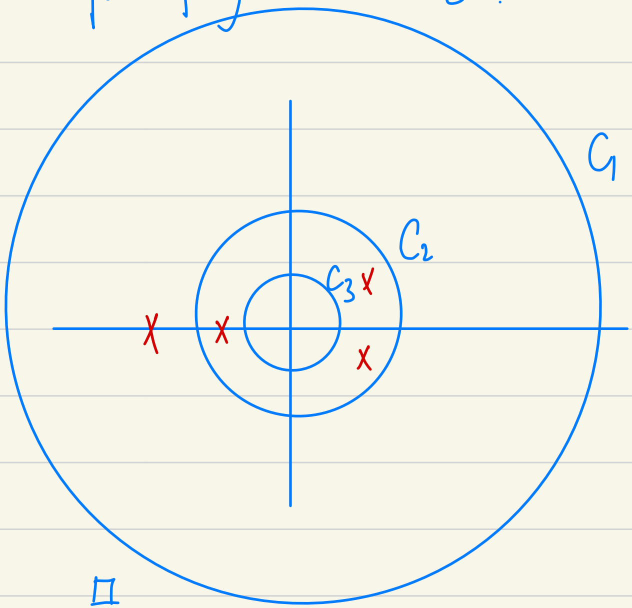
As, $f(z)$ has a root of multiplicity 3

Rouché's thm $\Rightarrow$ 3 roots of $f+g$ inside C_2.

- Let $C_3 : |z|=1$. $f(z)=6$, $g(z)=z^4+3z^3$.
  On $|z|=1$, $|g(z)| \le 4 < 6 = |f(z)|$
  Rouché's thm $\Rightarrow$ no root for $f+g$ inside C_3.

- Finally, Complex roots come up in conjugate pairs.

$\Rightarrow$ 2 real roots and 2 complex roots



Indeed, $P(-1)=4>0$
$P(-2)=-2<0$
$P(-3)=6>0$

$\square$

Yet another pf of Fundamental Theorem of Algebra

Assume $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ $(a_n \neq 0)$ degree n polynomial.

Let $f(z) = a_n z^n$. $g(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$.

Consider circle $|z| = R$, where R large enough $\left(> \max\left\{ \frac{|a_0| + \cdots + |a_{n-1}|}{|a_n|}, 1 \right\} \right)$

Then on $|z| = R$,
$$\begin{aligned} |g(z)| &\le |a_0| + |a_1| R + \cdots + |a_{n-1}| \cdot R^{n-1} \\ &\le (|a_0| + \cdots + |a_{n-1}|) \cdot R^{n-1} \\ &< |a_n| \cdot R^n = |f(z)| \end{aligned}$$

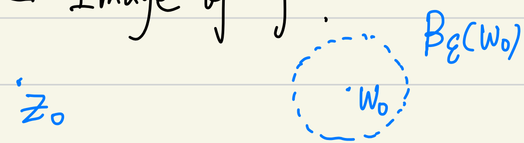
Rouché's thm $\Rightarrow$ $f+g$ and f has same number of zeroes $(=n)$ inside $|z| = R$ $\square$

Def: A map is open if it maps open sets to open sets.

For the complex function $w = f(z)$, this means if $f(z_0) = w_0$,

then there's $\varepsilon > 0$, s.t. for any $|w - w_0| < \varepsilon$, there is z s.t. $f(z) = w$.

In other words, $B_\varepsilon(w_0) := \{ |w - w_0| < \varepsilon \} \subset$ Image of f.

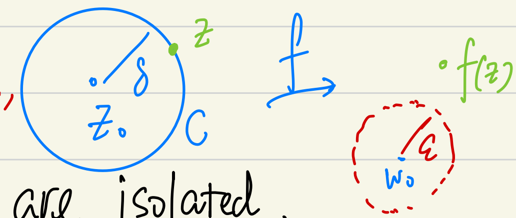


Thm (Open mapping theorem): If f is holomorphic and non-constant on an open domain D, then f is open.

to be specified later

Pf: For w "near" w_0, let $g(z) = f(z) - w$. Want to show $g(z)$ has a zero.

Write $g(z) = \underbrace{(f(z) - w_0)}_{F(z)} + \underbrace{(w_0 - w)}_{G(z)\ \text{"Const"}}$



Since $F(z)$ has a zero at z_0, and zeroes are isolated, there is $\delta > 0$ s.t. $F(z) \neq 0$ in $0 < |z - z_0| \leq \delta$.

In particular, on the circle C: $|z - z_0| = \delta$, assume $|F(z)| > \varepsilon > 0$

$= |f(z) - w_0|$.

Then for any $|w - w_0| < \varepsilon$, $|F(z)| > \varepsilon > |G(z)|$.

Rouché thm $\Rightarrow$ $g = F + G$ and F have same # zeroes inside C. (≥ 1)

$\square$

Corollary. (Maximal modulus Principle)

If f is holomorphic, non-constant on D, then $|f|$ cannot attain a maximum in the interior of D.