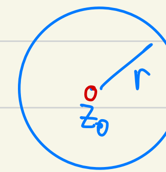
Note 6

§ Singularities

Def: · A function $f(z)$ is singular at z_0 if it is NOT holomorphic at z_0.

· The singularity z_0 is an isolated singularity if f is holomorphic on $0 < |z - z_0| < r$ for some $r > 0$.



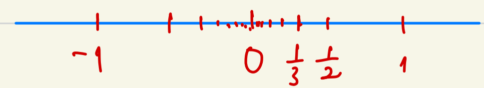
Ex: · $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ isolated singularities at $0, \pm i$

· $f(z) = e^{\frac{1}{z}}$ . . . — — — 0

· $f(z) = \dfrac{1}{\sin\left(\frac{\pi}{z}\right)}$ has singularities at $z=0$. and $\dfrac{1}{n}$, $n = \pm 1, \pm 2, \cdots$

isolated

NOT isolated!



- Classification of isolated singularities

Suppose $f(z)$ has an isolated singularity at z_0, then holomorphic on $0<|z-z_0|<r$

$\Rightarrow$ have Laurent series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$.

Def: · If $b_n=0$ for all n, then z_0 is called a removable singularity

(If we define $f(z_0)=a_0$ then f is analytic on disk $|z-z_0|<r$)

· If $\exists k$ s.t. $b_k \neq 0$ and $b_n=0$ $\forall n>k$, then z_0 is called a pole of order k.

In particular, pole of order 1 is called a simple pole.

($(z-z_0)^k \cdot f(z)$ has a removable singularity at z_0.)

· If there are inf. many $b_k \neq 0$, then z_0 is called an essential singularity.

Example:

- $f(z) = \frac{z+1}{z} = 1 + \frac{1}{z} \Rightarrow z = 0$ simple pole

- $f(z) = \frac{z+1}{z^3(z^2+1)}$. Singularities 0 pole of order 3

  $\pm i$ simple pole

  At $z_0 = 0$, $f(z) = \frac{1}{z^3} \cdot \frac{z+1}{z^2+1} =: g(z)$ analytic at 0, $g(0) = 1$

  $= 1 + a_1 z + a_2 z^2 + \cdots$

  $= \frac{1}{z^3} + \frac{a_1}{z^2} + \frac{a_2}{z} + \cdots \Rightarrow$ pole of order 3

- $f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{6} + \frac{z^4}{5!} - \cdots \Rightarrow z = 0$ removable singularity

  $\sin z = z - \frac{z^3}{6} + \frac{z^5}{5!} - \cdots$

- $f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \frac{1}{3! z^3} + \cdots$

  $\Rightarrow z = 0$ is an essential singularity.

- Behavior near singularities.

(1) z_0 is a removable singularity $\Leftrightarrow$ f is bounded near z_0.

pf: $\Rightarrow$: $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic hence continuous, so bounded in a compact set.

$\Leftarrow$: Recall that f has a Laurent series $\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ in $0 < |z-z_0| < r$

with coefficients $b_n = \frac{1}{2\pi i} \int_C f(z) \cdot (z-z_0)^{n-1} dz$.

Let C_ε circle $|z-z_0| = \varepsilon < r_0$. then $|b_n| \leq \frac{1}{2\pi} \max|f| \cdot \varepsilon^{n-1} \cdot \text{length}(C_\varepsilon) = \max|f| \cdot \varepsilon^n$.

ε can be arbitrarily small $\Rightarrow b_n = 0$ $\longrightarrow 0$ when $\varepsilon \to 0$

$\square$

(2) z_0 is a pole of order k $\iff$ $f(z) \sim \frac{1}{(z-z_0)^k}$ near z_0

In particular, $\boxed{\lim_{z\to z_0} |f(z)| = \infty}$.

pf: z_0 pole of order k $\iff$ removable singularity for $(z-z_0)^k f(z)$

Note $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{k} \frac{b_n}{(z-z_0)^n}$

$$= \frac{1}{(z-z_0)^k}\Big(\underbrace{b_k + b_{k-1}(z-z_0) + \cdots + b_1 (z-z_0)^{k-1} + a_0 (z-z_0)^k + \cdots}_{\approx\, b_k \text{ for } z \approx z_0.}\Big)$$

17

(3) For z_0 essential singularity, we have:

Picard Theorem: If $f(z)$ has an essential singularity at z_0, then $\forall$ nghd of z_0, f takes all complex values infinitely many times, with the possible exception of one value.

Ex: $f(z) = e^{\frac{1}{z}}$ takes every value (∞-times) except 0.

pf: Let $z = x+iy$, $c = \rho \cdot e^{i\theta} \neq 0$

Solve the equation $e^{\frac{1}{z}} = c \iff e^{\frac{x-iy}{x^2+y^2}} = \rho e^{i\theta} \iff \begin{cases} \frac{x}{x^2+y^2} = \ln\rho \\ -\frac{y}{x^2+y^2} = \theta + 2n\pi, & n \in \mathbb{Z} \end{cases}$

$\Rightarrow x_n = \frac{\ln\rho}{(\ln\rho)^2 + (-\theta+2n\pi)^2}$, $y_n = \frac{-\theta+2n\pi}{(\ln\rho)^2+(-\theta+2n\pi)^2}$. $\lim\limits_{n\to\infty} z_n = 0$.

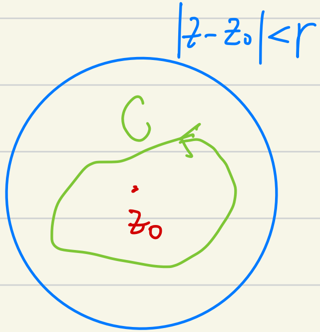
§ Residues

Def: Suppose z_0 is isolated singularity.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad \text{on} \quad 0<|z-z_0|<r$$

The residue of f at z_0 is b_1.

Denoted: $\mathrm{Res}(f, z_0) = \underset{z=z_0}{\mathrm{Res}}\, f = b_1$.



Alternative characterization: $\boxed{\mathrm{Res}(f, z_0) = b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz}$

Check: $\displaystyle\int_C (z-z_0)^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$

Example: (1) f holomorphic at z_0, $\mathrm{Res}(f, z_0) = 0$.

(2) $f(z) = \frac{\sin(z)}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \cdots\right) = 1 - \frac{z^2}{3!} + \cdots$

removable singularity at 0, $\mathrm{Res}(f, 0) = 0$.

(3) $f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots$

$\mathrm{Res}(f, 0) = 1$.

(4). $f(z) = \frac{1}{z(z^2+1)} = \frac{1}{z}(1 - z^2 + z^4 - \cdots)$ on $0 < |z| < 1$

$\mathrm{Res}(f, 0) = 1$

(5) $f(z) = \frac{1}{\sin(\frac{\pi}{z})}$. $z_0 = 0$ is non-isolated singularity. residue not defined!

Residue at Poles:

Prop 1: Suppose z_0 is a simple pole of $f(z)$. Let $g(z) = (z - z_0) \cdot f(z)$

then $\boxed{Res(f, z_0) = g(z_0)}$

Pf 1: Laurent series $f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$

$$\Rightarrow \quad g(z) = b_1 + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots$$

□

Pf 2: $Res(f, z_0) = \int_C f(z)\, dz = \int_C \frac{g(z)}{z - z_0} dz = g(z_0)$

Cauchy Integral Formula

□.

Prop 2: If $g(z)$ has a simple zero at z_0, then $f(z)=\frac{1}{g(z)}$ has a simple pole at z_0

and $\boxed{\operatorname{Res}(f, z_0) = \frac{1}{g'(z_0)}}$

Pf: Recall for $g(z) = \cancel{a_0}^{\,0} + a_1(z-z_0) + \cdots$, z_0 is a zero of order k if $a_k \neq 0$ and $a_n = 0$ $\forall n < k$.

In particular, $a_1 \neq 0$ for a simple zero, and $g'(z_0) = a_1$.

Also, $f = \frac{1}{g} = \frac{1}{z-z_0} \cdot \frac{1}{a_1 + a_2(z-z_0) + \cdots} = \frac{1}{z-z_0}\left(\underset{=\frac{1}{a_1}}{c_0} + \underset{=-\frac{a_2}{a_1^2}}{c_1}(z-z_0) + c_2(z-z_0)^2 + \cdots\right)$

$\Rightarrow \operatorname{Res}(f, z_0) = \frac{1}{a_1} = \frac{1}{g'(z_0)}$. $\square$

Ex: • $f(z) = \dfrac{1}{z(z^2+1)}$ $\quad \text{Res}(f,0) = 1$

• $f(z) = \dfrac{2+z+z^2}{(z-2)(z-3)(z-4)(z-5)}$ $\quad \text{Res}(f,2) = \dfrac{2+2+4}{(-1)\cdot(-2)(-3)} = -\dfrac{4}{3}.$

Ex. • $f(z) = \dfrac{1}{\sin z}$: $\quad g(z) = \sin z$ has simple zeroes at $n\pi \quad n \in \mathbb{Z}$

$(\sin'(n\pi) = \cos n\pi \neq 0)$

$\Rightarrow \text{Res}(f, n\pi) = \dfrac{1}{g'(n\pi)} = \dfrac{1}{\cos(n\pi)} = (-1)^n.$

• $f(z) = \dfrac{1}{z^6+1} = \dfrac{1}{\prod_{k=1}^{6}(z-z_k)}$ $\quad z_k^6 = -1 \iff z_k = e^{i\frac{\pi+2k\pi}{6}} \quad k=1,\cdots,6$

$\Rightarrow \text{Res}(f, z_k) = \dfrac{1}{(z^6+1)'\big|_{z=z_k}} = \dfrac{1}{6z_k^5} = -\dfrac{z_k}{6}.$

- Suppose z_0 is a pole of order k of $f(z)$. Let $g(z) = (z-z_0)^k \cdot f(z)$

then $\boxed{\mathrm{Res}(f, z_0) = \dfrac{g^{(k-1)}(z_0)}{(k-1)!}}$

Rmk: apply to pole of order $\le k$

Pf: Laurent series $f(z) = \dfrac{b_k}{(z-z_0)^k} + \cdots \dfrac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$

$$\Rightarrow g(z) = b_k + \cdots + b_1(z-z_0)^{k-1} + \cdots$$

$$\mathrm{Res}(f, z_0) = b_1 = \frac{g^{(k-1)}(z_0)}{(k-1)!}$$

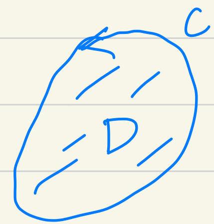
$\square$

Ex: $f(z) = \dfrac{1}{z(z^2+1)(z-2)^2}$

$g(z) = (z-2)^2 \cdot f(z) = \dfrac{1}{z(z^2+1)}$ $\quad g'(2) = \left.\dfrac{-(3z^2+1)}{z^2(z^2+1)^2}\right|_{z=2} = \dfrac{-13}{4 \cdot 25} = -\dfrac{13}{100}$

Hence $\mathrm{Res}(f, 2) = \dfrac{g'(2)}{1!} = -\dfrac{13}{100}$.

§ Cauchy Residue Theorem.



Recall: In multi-variable Calculus.

- When $\vec{F} = \nabla f$, then $\int_C \vec{F} \cdot d\vec{r} = 0$ ← curl $\vec{F} = 0$.

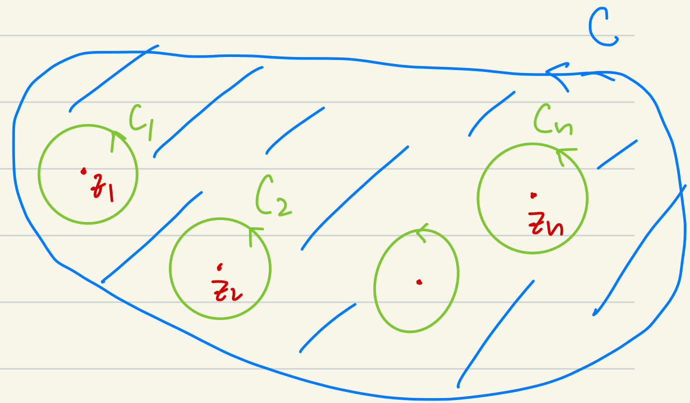
- For general $\vec{F}$, Green thm: $\int_C \vec{F} \cdot d\vec{r} = \int_D \text{curl}\, \vec{F} \cdot dA$

- For Complex integral, Cauchy-Goursat thm: $f(z)$ holomorphic then $\int_C f(z)\, dz = 0$

Analogue of Curl & Green thm : Residue & Cauchy Residue Theorem.

$$\int_C f(z)\, dz \sim \sum_{i=1}^{n} \text{Res}(f, z_i) \leftarrow \text{"} \int_D \text{Res}(f,z)\, dA \text{"}$$ if we let $\text{Res}(f, z) = 0$ for holomorphic pt.

Cauchy Residue Thm: Let C be a simple closed curve and f holomorphic inside C except for a finite number of singular pts $z_1, \cdots, z_n$.



Then $$\boxed{\int_C f(z)\,dz = 2\pi i \cdot \sum_{i=1}^{n} \mathrm{Res}(f, z_i)}$$

Pf: Cauchy-Goursat Thm

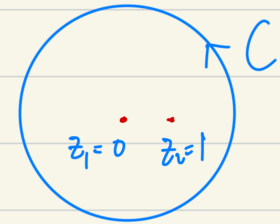
$$\Rightarrow \int_C f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz$$

$$= 2\pi i \sum_{i=1}^{n} \mathrm{Res}(f, z_i)$$

Definition of Residue

$\square$

Ex: • $\displaystyle\int_C \frac{4z-5}{z\cdot(z-1)}\,dz.$ $\quad C: |z|=2.$

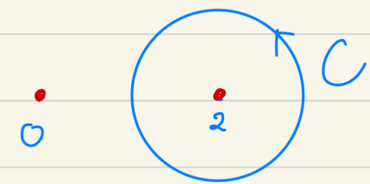


Residue thm $= 2\pi i\left(\mathrm{Res}(f,0)+\mathrm{Res}(f,1)\right)$

$= 2\pi i\left(5+(-1)\right)$

$= 8\pi i$

• $\displaystyle\int_C \frac{1}{z(z-2)^4}\,dz$ $\quad C: |z-2|=1$



$= 2\pi i\,\mathrm{Res}(f,2)$

2 is a pole of order 4

$\Rightarrow \mathrm{Res}\, f = \dfrac{g^{(3)}(2)}{3!} = -\dfrac{1}{16}$

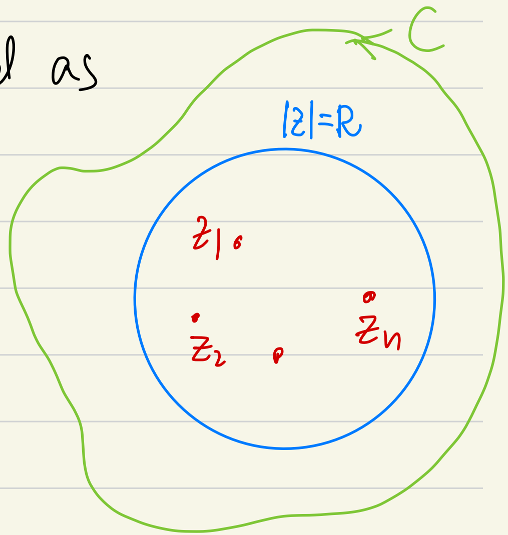
$g = f\cdot(z-2)^4 = \dfrac{1}{z}$

$g^{(3)}(z) = -3!\,\dfrac{1}{z^4}$

$= -\dfrac{\pi i}{8}.$

Def: Suppose f is analytic in $\mathbb{C}$ except for a finite number of singularities. Let C be a large enough simple closed curve that contains all singularities. Then the residue of f at infinity is defined as

$$\boxed{\operatorname{Res}(f, \infty) := -\frac{1}{2\pi i}\int_C f(z)\,dz}$$



Cauchy Residue Thm

$$\Rightarrow \quad \operatorname{Res}(f, \infty) = -\sum_{i=1}^{n} \operatorname{Res}(f, z_i).$$

Suppose all singularities in $|z| < R$, then f holomorphic in $R < |z| < \infty$

$\Rightarrow$ Laurent series: $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$

$\Rightarrow$ $b_1 = \frac{1}{2\pi i} \int_C f(z)\,dz = -\operatorname{Res}(f, \infty)$

Theorem: $\boxed{\operatorname{Res}(f, \infty) = -\operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right)}$.

pf: $\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \cdot \left(\sum_{n=1}^{\infty} b_n z^n + \sum_{n=0}^{\infty} a_n z^{-n}\right)$ Converges for $R < \left|\frac{1}{z}\right| < \infty$

$\Leftrightarrow$ $0 < |z| < \frac{1}{R}$

$\Rightarrow$ $\operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right) = b_1 = -\operatorname{Res}(f, \infty)$

□

Ex: $f(z) = \dfrac{4z-5}{z(z-1)}$.

Earlier, we computed $\displaystyle\int_C f(z)\,dz = 8\pi i$ $C: |z|=2$

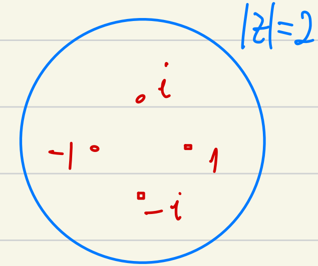
Recompute this using residue at ∞.

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \cdot \frac{\frac{4}{z}-5}{\frac{1}{z}\cdot\left(\frac{1}{z}-1\right)} = \frac{1}{z^2} \frac{(4-5z)\cdot z \cdot z}{z\cdot(1-z)} = \frac{4-5z}{z(1-z)}$$

$\Rightarrow \operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right) = 4$. $\Rightarrow \operatorname{Res}(f, \infty) = -4$

$\Rightarrow \displaystyle\int_C f(z)\,dz = -2\pi i \cdot \operatorname{Res}(f,\infty) = 8\pi i$.

- $\displaystyle\int_C \frac{z^3}{z^4-1}\,dz \qquad C: |z|=2.$



Let $f(z) = \dfrac{z^3}{z^4-1}$

Then $\displaystyle \frac{1}{z^2}\cdot f\left(\frac{1}{z}\right) = \frac{1}{z^2}\cdot\frac{\left(\frac{1}{z}\right)^3}{\left(\frac{1}{z}\right)^4-1} = \frac{1}{z^2}\cdot\frac{z}{1-z^4} = \frac{1}{z(1-z^4)}.$

$\Rightarrow \operatorname{Res}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right), 0\right) = 1 \qquad \Rightarrow \operatorname{Res}(f,\infty) = -1$

$$\Rightarrow \int_C \frac{z^3}{z^4-1}\,dz = -2\pi i\cdot \operatorname{Res}(f,\infty) = 2\pi i.$$

$\square$