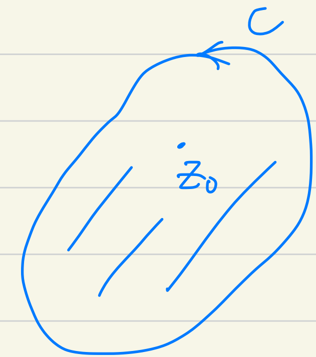
Note 4

§ Cauchy Integral Formula

Theorem: Suppose C is a simple closed curve and $f(z)$ holomorphic everywhere inside and on C.



Then for any z_0 inside C,

$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz}$$

Equivalently, $f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw$

Rmk: Values of f on the boundary C determine values of f inside!

<u>Ex:</u> • $f(z) = 1$: $1 = \frac{1}{2\pi i} \int_C \frac{1}{z - z_0} dz$

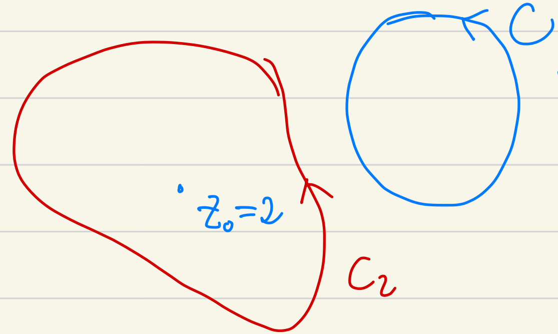
• Cauchy-Goursat Thm as special case:

Let $f(z)$ holomorphic, $g(z) = f(z) \cdot (z - z_0)$ also holomorphic

Then $0 = g(z_0) = \int_C \frac{g(z)}{z - z_0} dz = \int_C f(z) dz$.

· Compute $\displaystyle\int_C \frac{e^{z^2}}{z-2}\,dz$

Two cases:



(i) C_1 does not enclose $z_0=2$.

Then $\dfrac{e^{z^2}}{z-2}$ is holomorphic on and inside C $\Rightarrow$ $\displaystyle\int_{C_1} \frac{e^{z^2}}{z-2}\,dz = 0$

(ii) C_2 encloses $z_0=2$. Let $f(z)=e^{z^2}$

Cauchy's Integral Formula $\Rightarrow$ $\displaystyle f(2)=\frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z-2}\,dz$

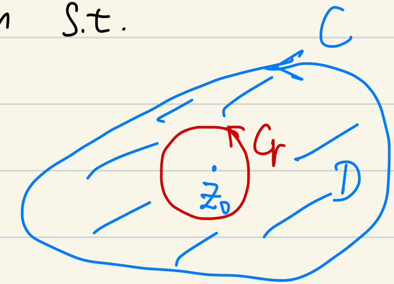
$$\Rightarrow \int_{C_2} \frac{e^{z^2}}{z-2}\,dz = 2\pi i\cdot e^4.$$

Lemma (Refinement of Cauchy Thm): Suppose C is a simple closed curve, $z_0 \in D$ and g is a function s.t.

(1) holomorphic on $D - \{z_0\}$

(2) continuous at z_0

Then $\int_C g(z)\,dz = 0$.



Pf: Let C_r = circle of radius r around z_0 inside C.

Then $\int_C g(z)\,dz = \int_{C_r} g(z)\,dz$

Since $g(z)$ cont. $\Rightarrow |g(z)| \leq M$ inside C_r

$\Rightarrow \left| \int_{C_r} g(z)\,dz \right| \leq M \cdot (\text{length of } C_r) = M \cdot 2\pi r \longrightarrow 0$ when $r \to 0$.

$\Rightarrow \int_{C_r} g(z)\,dz = 0$

□

Pf of Cauchy Integral formula:

Let $g(z) := \begin{cases} \dfrac{f(z)-f(z_0)}{z-z_0} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}$

Note: $g(z)$ holomorphic on $D-\{z_0\}$ as $f(z)$ holomorphic on z

Also, $\lim\limits_{z \to z_0} g(z) = f'(z_0) = g(z_0) \Rightarrow g$ continuous at z_0

Lemma $\Rightarrow \int_C g(z)\,dz = 0 \iff \int_C \dfrac{f(z)-f(z_0)}{z-z_0}\,dz = 0$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0}\,dz = \int_C \frac{f(z_0)}{z-z_0}\,dz = f(z_0)\cdot \int_C \frac{1}{z-z_0}\,dz = 2\pi i \cdot f(z_0)$$

$\square$

Thm (Cauchy Integral formula for derivatives)

Same hypothesis on $f(z)$ and C, then $\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.}$

Equivalently. $f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}\,dw$

(Intuitive) Proof: Recall $f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw$

Take derivative $f'(z) = \frac{d}{dz}\left(\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw\right)$ concern? $\frac{d}{dz}\int \overset{?}{=} \int \frac{d}{dz}$

$= \frac{1}{2\pi i}\int_C \frac{d}{dz}\left(\frac{f(w)}{w-z}\right)dw$

$= \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}\,dw$ General formula by induction on n.

(Rigorous) Proof:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Using the integral rep.:

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z-\Delta z}\,dw - \int_C \frac{f(w)}{w-z}\,dw}{\Delta z}$$

$$= \frac{1}{2\pi i\,\cancel{\Delta z}} \int_C \frac{\cancel{\Delta z}\cdot f(w)}{(w-z-\Delta z)(w-z)}\,dw$$

In the limit $\Delta z \to 0$, $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}\,dw$ // when $\Delta z \to 0$.

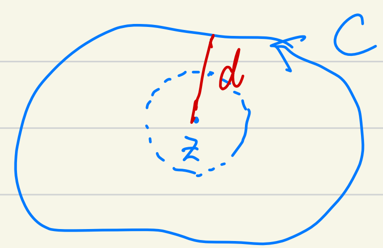
"Taking the limit under integral": $\lim_{\Delta z \to 0} \int \overset{?}{=} \int \lim_{\Delta z \to 0}$

Justification: $$\left| \int_C \frac{f(w)}{(w-z-\Delta z)\cdot(w-z)}\,dw - \int_C \frac{f(w)}{(w-z)^2}\,dw \right|$$

$$= \left| \int_C \frac{\Delta z \cdot f(w)}{(w-z)^2\cdot(w-z-\Delta z)}\,dw \right|$$

$$\leq \frac{|\Delta z|\cdot M}{(d-|\Delta z|)\,d^2}\,(\text{length of } C) \longrightarrow 0 \quad \text{as } \Delta z \longrightarrow 0.$$

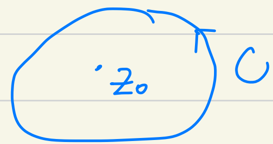
(Let $M = \max |f(w)|$ on C
$d = \min |w-z|$ on C
$\Rightarrow |w-z-\Delta z| \geq d - |\Delta z|$)



$\square$

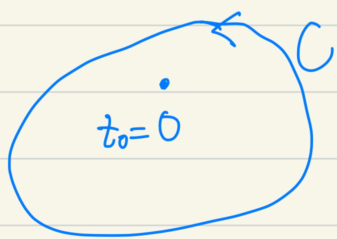
$f(z)=1$

Example: · $\displaystyle\int_C \frac{dz}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!}\cdot f^{(n)}(z_0) = \begin{cases} 2\pi i & n=0. \\ 0 & n=1,2,\dots \end{cases}$



· $\displaystyle\int_C \frac{e^{2z}}{z^4}\, dz = \frac{2\pi i}{3!}\; f^{(3)}(z)\Big|_{z=0.} = \frac{8\pi i}{3}.$

$f(z) = e^{2z}. \qquad (e^{2z})''' = 8\cdot e^{2z}$



§ Important Consequences of Cauchy Integral formula.

1) ∞ - Differentiability
2) Fundamental Theorem of Algebra
3) Mean Value Property and Maximal Principle

1) Existence of derivatives.

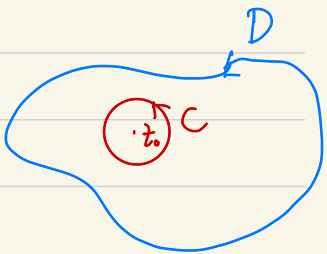
No Simply-Connected assumption

Thm Suppose f is holomorphic on a region D. i.e., $f'(z)$ exists for $z \in D$

Then f has derivative of all orders. i.e., $f^{(n)}(z)$ exists $\forall n$.

Pf: For all z_0, take a small disk around $z_0 \subset D$.

Apply Cauchy Integral formula $\Rightarrow$ $f^{(n)}(z_0)$ in terms of integral over C.



Thm: (Morera's thm): Suppose f continuous on a domain D and

$$\int_C f(z)\,dz = 0 \qquad \text{for all closed curve in } C$$

then f holomorphic on D.

Pf: Equivalence of path-indep $\Rightarrow$ f has antiderivative F: $F'(z) = f(z)$, $\forall z \in D$

Then F has derivatives of all orders $\Rightarrow$ So is f.

$\square$

harmonic

Thm: If $f(z)$ holomorphic, then $u(x,y)$ and $v(x,y)$ are smooth f^n's i.e., $\exists$ partial der. of all orders.

Pf: $f'(z) = u_x + i v_x = v_y - i u_y$, then $f''(z) = u_{xx} + i v_{xx} = v_{yx} - i u_{yx}$, etc. ... $\square$

Rmk: Related results for harmonic functions in real variable functions

$H(x,y)$ harmonic $\iff H_{xx} + H_{yy} = 0$

More generally, $H(x_1, \ldots, x_n)$ smooth $\iff \sum_{i=1}^{n} \frac{\partial^2 H}{\partial x_i^2} = 0$

Thm: A harmonic function is smooth.

2) Cauchy's Inequality / estimate

Theorem: Suppose $f(z)$ is holomorphic in $C_R := \{|z-z_0| = R\}$. Let $M_R = \max_{z \in C_R} |f(z)|$

Then $\boxed{|f^{(n)}(z_0)| \leq \frac{n!\, M_R}{R^n}}$

pf: Recall $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$

Note $\left| \frac{f(z)}{(z-z_0)^{n+1}} \right| \leq \frac{M_R}{R^{n+1}} \quad \forall z \in C_R$

$$\Rightarrow |f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} \cdot \underbrace{(\text{length of } C_R)}_{= 2\pi R} = \frac{n!\, M_R}{R^n}$$

□

Thm (Liouville's Thm) If f is entire and bounded in the complex plane, then f is constant.

Pf. Suppose $|f(z)| < M \quad \forall z \in \mathbb{C}$.
(bounded)

Apply Cauchy's Ineq $\Rightarrow |f'(z_0)| \leq \frac{M}{R} \longrightarrow 0$ when $R \to \infty$

But R can be as large as we like $\Rightarrow |f'(z_0)| = 0$.
(entire)

$\Rightarrow f$ is constant.

$\square$

Corollary (Fundamental Thm of Algebra)

Any polynomial $P(z)=a_0+a_1z+\cdots+a_nz^n$, $a_n\neq 0$ of degree n has exactly n roots.

Pf. (1) P of degree $n\geq 1$ has at least one root:

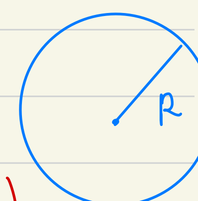
Pf by Contradiction: Suppose $P(z)$ does not have a root. Then

- $f(z)=\frac{1}{P(z)}$ is entire.
- $f(z)$ is bounded $\because$ $\left|\frac{1}{P(z)}\right|$ goes to 0 as $|z|$ goes to ∞

  ($|P(z)|\sim a_n|z|^n \longrightarrow \infty$. refer to textbook for details)

  - $\left|\frac{1}{P(z)}\right|$ is bounded in any disk $|z|\leq R$

    Compact domain



Liouville's Thm $\Rightarrow f(z)$ const $\Rightarrow P(z)$ Const. Contradiction!

(ii) P has exactly n roots. Let z_1 be one zero,

Factor $P(z) = (z-z_1)\cdot Q(z)$. $Q(z)$ has degree $n-1$.

If $n-1 \geq 1$, $Q(z)$ has at least one root z_2, then factor out $z-z_2$.
and continue this process.

At the end, can write $P(z) = a_n(z-z_1)\cdots(z-z_n)$ $\square$

Rmk: Non-Constructive proof, didn't give an algorithm for finding roots.

3) Maximum (modulus) principle:

Recall Extremal Value Problem for real function $y=f(x)$.

max/min $f(x)$ at critical points: $f'(x)=0$

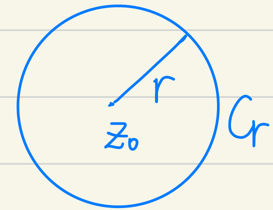
Different story for complex function $w=f(z)$:

Roughly speaking, $|f(z)|$ has no (relative) maximum in the interior of D

Consequently, $\max_{z\in D}|f(z)|$ can only be found on boundary of D.

Thm (Mean Value Property): Suppose $f(z)$ is holomorphic on closed disk $|z-z_0|\le r$.

Then $\boxed{f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta}$



"mean value" on C_r

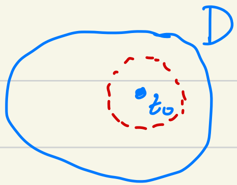
Pf: Cauchy's Integral formula $\Rightarrow$ $f(z_0)=\frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{z-z_0}\,dz$

Parametrize C_r: $z(\theta)=z_0+re^{i\theta}$, $0\le\theta\le 2\pi$. $dz = z'(\theta)\,d\theta = rie^{i\theta}\,d\theta$

$z-z_0 = re^{i\theta}$

$$\Rightarrow f(z_0)=\frac{1}{2\pi \cancel{i}}\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{\cancel{re^{i\theta}}}\cdot\cancel{r}\cdot\cancel{i}\cdot\cancel{e^{i\theta}}\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta.$$



Thm (Maximum modulus Principle, Local Version)

Suppose f is holomophic in a domain D, and z_0 is an interior point in D.

If $|f(z)|$ has a relative maximum at z_0, then $f(z)$ is constant in a nghd of z_0.

Pf: Suppose $|f(z_0)|$ rel. maximum. So there exists a small circle C_r: $|z-z_0|=r$

s.t. $|f(z_0)| \geq |f(z)|$ $\forall z$ inside C_r.

By mean value property and triangle inequality:

$$|f(z_0)| \overset{\text{mean value prop.}}{=} \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta\right|$$

$$\overset{\text{triangle ineq}}{\leq} \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|\,d\theta \qquad (1)$$

$$\overset{\text{max. assumption}}{\leq} \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,d\theta \qquad (2)$$

$$= |f(z_0)|$$

Thus, all inequalities must be equalities.

Ineq (1) $\Rightarrow$ $f(z_0+re^{i\theta})$ same argument

Ineq (2) $\Rightarrow$ $|f(z_0+re^{i\theta})| = |f(z_0)|$ i.e., same modulus.

Hence, f const along C_r. As $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{f(z_0+re^{i\theta})}_{\text{const}}\,d\theta \Rightarrow f(z) = f(z_0)$ on C_r

$\square$

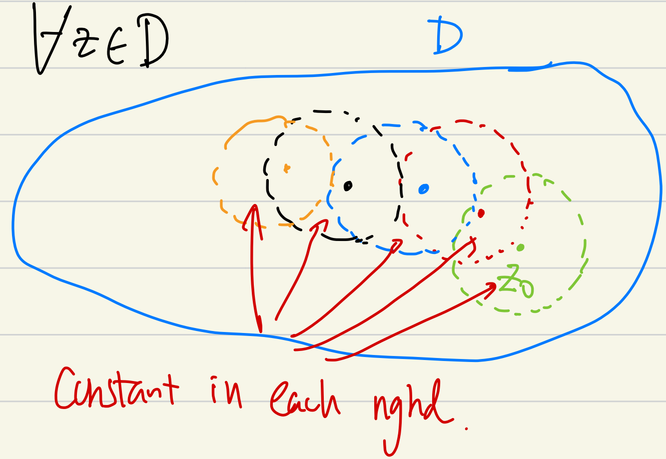
Thm: (Maximum modulus principle, Global Version)

If f is non-constant and holomorphic on D,

then $|f(z)|$ has no global maximum value inside D. i.e, there's no interior z_0

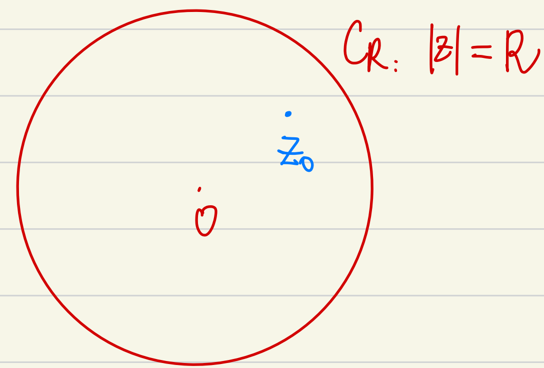
s.t. $|f(z)| \le |f(z_0)| \quad \forall z \in D$

"pf" by picture



Corollary: Suppose $f(z)$ is entire. If $\lim_{z\to\infty} f(z) = 0$, then $f(z) \equiv 0$.

pf: Fix z_0. Let C_R large circle containing z_0.



Maximum modulus principle $\Rightarrow \max_{|z|\le R} |f(z)| = \max_{z\in C_R} |f(z)|$

In particular, $|f(z_0)| \le \max_{z\in C_R} |f(z)| =: M_R$

This is true for all large enough R,

As $\lim_{z\to\infty} f(z) = 0$. $\lim_{R\to\infty} M_R = 0$. $\Rightarrow f(z_0) = 0$.

$\square$

Rmk. (1) No "minimum modulus principle" in general: may have $f(z_0) = 0$.

On the other hand, if $f \neq 0$ in D, minimum modulus principle holds.

(2) Harmonic functions also satisfy:

- Mean value property
- maximum principle
- Liouville's theorem.