Note 3

§ Definite Integral

Let $w(t) = \underbrace{u(t)}_{Re} + i \cdot \underbrace{v(t)}_{Im} \in \mathbb{C}$. Complex-valued real-variable function.

$$\int_a^b w(t)\,dt := \underbrace{\int_a^b u(t)\,dt}_{Re} + i \underbrace{\int_a^b v(t)\,dt}_{Im}$$

Fund. Thm of Calculus: Suppose $W(t) = U(t) + i\,V(t)$ anti-derivative $W(t)$

s.t. $W'(t) = U'(t) + i\,V'(t) = u(t) + i\,v(t) = w(t)$

then $\int_a^b w(t)\,dt = W(b) - W(a)$.

Ex: $\int_0^{\frac{\pi}{4}} e^{it}\,dt$. Note $\frac{d}{dt}(e^{it}) = i \cdot e^{it}$. $\Rightarrow \frac{d}{dt}\left(\frac{e^{it}}{i}\right) = e^{it}$

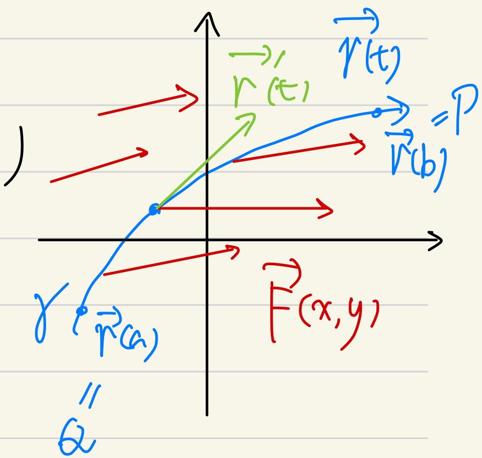
$$\Rightarrow \int_0^{\frac{\pi}{4}} e^{it}\,dt = \frac{e^{i\cdot\frac{\pi}{4}}}{i} - \frac{1}{i} = -i\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - 1\right) = \frac{1}{\sqrt{2}} + \left(1 - \frac{1}{\sqrt{2}}\right) i$$

§ Complex line Integral $\int_\gamma f(z)dz$

$\approx$ line integrals in 2-variable calculus

Quick Review of MATH2020:

1) Line Integral Definition:



- Parametrized Curves in the plane $\mathbb{R}^2$. $\vec{r}(t) = (x(t), y(t))$
  tangent vector $\vec{r}'(t) = (x'(t), y'(t))$ $\quad a \le t \le b$

- Vector field $\vec{F}(x,y) = \langle M(x,y), N(x,y) \rangle$

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_\gamma M dx + N dy .$$

dot product

Physically, work

2) Fundamental Theory for Line Integrals

- Function $f(x,y)$: $\quad \nabla f := \langle f_x, f_y \rangle$ gradient vector field

For $\vec{r}(t) = (x(t), y(t))$, Chain rule $\Rightarrow$ $\boxed{\dfrac{d f(\vec{r}(t))}{dt} = \nabla f \cdot \vec{r}\,'(t)}$

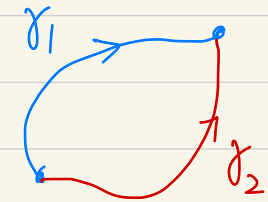
Theorem: If $\vec{F} = \nabla f$. then $\displaystyle\int_\gamma \vec{F} \cdot d\vec{r} = f(\underbrace{P}_{\gamma(b)}) - f(\underbrace{Q}_{\gamma(a)})$

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_a^b \nabla f \cdot \vec{r}\,'(t)\, dt = \int_a^b \frac{d f(\vec{r}(t))}{dt} dt = f(P) - f(Q)$$

Fund. Theory of calculus

3) Path Independence

Consequently, $\vec{F} = \nabla f$



$C = \gamma_2 \cup -\gamma_1$

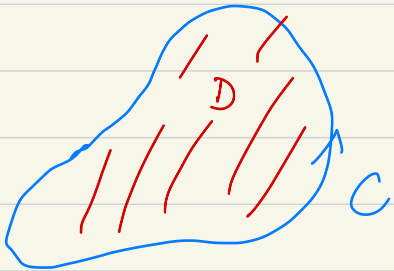
$\iff$ Path independence : $\int_{\gamma_1} \vec{F} \cdot d\vec{r} = \int_{\gamma_2} \vec{F} \cdot d\vec{r}$

$\iff$ $\boxed{\oint_C \vec{F} \cdot d\vec{r} = 0}$ for any closed path

4) Line integral over closed path

- Vector field $\vec{F}(x,y) = \langle M(x,y), N(x,y) \rangle$

$$\operatorname{curl} \vec{F} := N_x - M_y$$



Green's Theorem:

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F} \, dA}$$

or equivalently, $\oint_C M dx + N dy = \iint_D N_x - M_y \, dx dy$

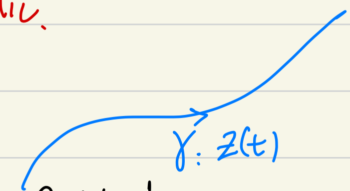
Note: If $\vec{F} = \nabla f = \langle f_x, f_y \rangle$, then $\operatorname{curl} \vec{F} = f_{yx} - f_{xy} = 0$. $\Rightarrow \oint_C \vec{F} \, d\vec{r} = 0$.

Back to Complex (line) integral.

Need not be holomorphic.

1). Given $f(z) = u(x,y) + i\,v(x,y)$ Complex function

$z(t) = x(t) + i\,y(t)$ Curve in complex plane. $a \leq t \leq b$



Def: Line / Path / Contour Integral $\int_\gamma f(z)\,dz = \int_a^b f(z(t)) \cdot z'(t)\,dt$

complex product

Ex: · $\int_{\gamma_1} z^2\,dz$. γ_1: straight line from 0 to $1+i$

· Parametrize $z(t) = (1+i)\,t \quad 0 \leq t \leq 1 \quad z'(t) = 1+i$

then $\int_{\gamma_1} z^2\,dz = \int_0^1 (1+i)^2 t^2 \cdot (1+i)\,dt = (1+i)^3 \cdot \left[\frac{t^3}{3}\right]_0^1 = \frac{(1+i)^3}{3}$

- $\int_{\gamma_2} z^2 dz$ γ_2: unit circle (Counterclockwise)

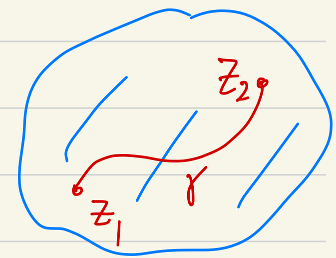
Parametrize $z(t) = e^{it}$ $0 \le t \le 2\pi$ $z'(t) = i \cdot e^{it}$

then $\int_{\gamma_2} z^2 dz = \int_0^{2\pi} e^{2it} \cdot ie^{it} dt = \int_0^{2\pi} i \cdot e^{3it} dt = \left[\frac{e^{3it}}{3}\right]\Big|_0^{2\pi} = 0$

- $\int_{\gamma_2} \bar{z} dz$ γ_2 unit circle.

$$\int_{\gamma_2} \bar{z} dz = \int_0^{2\pi} \overline{e^{it}} \cdot ie^{it} dt = \int_0^{2\pi} i dt = 2\pi i$$

2). Theorem (Fundamental Thm of Complex line Integrals)



If $F(z)$ is a holomorphic function on domain D, and γ is a curve in D from z_1 to z_2

then $\int_\gamma F'(z)\,dz = F(z_2) - F(z_1)$

Equivalently, If f has an anti-derivative F, i.e., $F' = f$

then $\int_\gamma f(z)\,dz = F(z_2) - F(z_1)$

Pf: Let $z(t)$ be the parametrized curve. $a \le t \le b$, $z(a) = z_1$, $z(b) = z_2$.

By chain rule, $\dfrac{d\,F(z(t))}{dt} = F'(z(t)) \cdot z'(t)$

Hence $\displaystyle\int_\gamma F'(z)\,dz = \int_a^b F'(z(t))\,z'(t)\,dt = \int_a^b \frac{d\,F(z(t))}{dt}\,dt$

$$= F(z(t))\Big|_a^b = F(z_2) - F(z_1)$$

Redo Ex: $\displaystyle\int_\gamma z^2\,dz = \frac{z^3}{3}\Big|_{z_1}^{z_2} = \begin{cases} \dfrac{(1+i)^3}{3} & \text{for straight line } \gamma_1 \\ 0 & \text{for unit circle } \gamma_2 \end{cases}$

Some useful Inequality:

Thm: (Triangle Inequality for definite integral)

Suppose $w(t) \in \mathbb{C}$ complex-valued f^n. then $\overset{r_0}{\overset{\shortparallel}{\left| \int_a^b w(t)\,dt \right|}} \le \int_a^b |w(t)|\,dt$

Pf: Suppose $\int_a^b w(t)\,dt = r_0 \cdot e^{i\theta_0}$ $\iff$ $\int_a^b w(t) \cdot e^{-i\theta_0}\,dt = r_0 \in \mathbb{R}$

Write $w(t) \cdot e^{-i\theta_0} = u(t) + i \cdot v(t)$ $\iff$ $\underbrace{\int_a^b u(t)\,dt}_{= r_0} + i \cdot \underbrace{\cancel{\int_a^b v(t)\,dt}}_{\to 0}$

Since $|u(t)| \le |w(t) \cdot e^{-i\theta_0}| = |w(t)|$ $\Rightarrow$ $\int_a^b u(t)\,dt \le \int_a^b |u(t)|\,dt$

$\le \int_a^b |w(t)|\,dt$

$\square$

Thm: (Triangle Inequality for Complex line integral)

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\cdot\underbrace{|dz|}_{|z'(t)|\,dt}. \quad \text{where } dz = z'(t)\,dt, \quad |dz| = |z'(t)|\,dt$$

pf. $$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b \underbrace{f(z(t))\,z'(t)}_{W(t)}\,dt\right| \le \int_a^b \underbrace{|f(z(t))|\,|z'(t)|}_{|W(t)|}\cdot dt = \int_\gamma |f(z)|\cdot|dz|$$ $\square$

Consequently, if $|f(z)| \le M$ on γ, then $$\int_\gamma |f(z)|\,|dz| \le M\cdot\underbrace{\int_\gamma |z'(t)|\,dt}_{\text{length of } \gamma}, \quad |z'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$$

$$\Rightarrow \boxed{\left|\int_\gamma f(z)\,dz\right| \le M\cdot(\text{length of } \gamma)}$$

3) Path Independence.

Thm: Suppose $f(z)$ is continuous in a domain D.
Then the following 3 things are equivalent:

(a) $f(z)$ has an antiderivative $F(z)$ in D, i.e., $F'(z) = f(z)$

(b) $\int_C f(z)\,dz = 0$ for any closed path.

(c) $\int_\gamma f(z)\,dz$ is path indep.

Rmk: For multi-var. calculus. $\vec{F} = \nabla f \iff \int_\gamma \vec{F}\cdot d\vec{r}$ path-indep $\iff \oint_C \vec{F}\cdot d\vec{r} = 0$

Pf: (a) $\Rightarrow$ (b): immediate from the Fundamental Thm of $\int_\gamma f(z)dz$

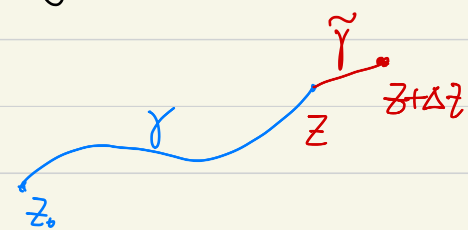
(b) $\Rightarrow$ (c). Standard argument

(c) $\Rightarrow$ (a): Pick a basepoint $z_0 \in D$.

Let $F(z) = \int_\gamma f(w)\,dw$, where γ arbitrary curve from z_0 to z.

- F well-defined since path-indep.

- Want $F'(z) = f(z)$.



By definition, $F(z+\Delta z) - F(z) = \int_{\tilde{\gamma}} f(w)\,dw$

Also, $\displaystyle\int_{\tilde{\gamma}} f(z)\,dw = \underbrace{f(z)}_{\text{Const.}} \int_{\tilde{\gamma}} 1\,dw \rightarrow w\Big|_{z}^{z+\Delta z} = \Delta z.$

$\displaystyle = f(z)\cdot \Delta z$

Hence, $\displaystyle\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{\tilde{\gamma}} \big(f(w)-f(z)\big)\,dw$

Since f is continuous, $\forall \varepsilon, \exists \delta$ s.t. $|f(w)-f(z)| < \varepsilon$ whenever $|w-z| < \delta$

Triangle Inequality $\Rightarrow \displaystyle\left|\frac{1}{\Delta z}\int_{\tilde{\gamma}} (f(w)-f(z))\,dw\right| \le \frac{1}{|\Delta z|}\cdot \varepsilon \cdot \underbrace{|\text{length of } \tilde{\gamma}|}_{=|\Delta z|} = \varepsilon \longrightarrow 0$

as $\delta \le |\Delta z| \longrightarrow 0$

$\displaystyle\Rightarrow \lim_{\Delta z \to 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = 0$ i.e., $F'(z) = f(z)$ □

Ex: · $\int_C \bar{z}\,dz$ is not path-indep $\Rightarrow$ $\bar{z}$ has no anti-derivative.

· $\int_C \frac{1}{z^2}\,dz$ C unit circle.

(1) Anti-derivative: $F(z) = -\frac{1}{z}$, $F'(z) = \frac{1}{z^2}$. defined over $\mathbb{C} - \{0\} \supset C$

$$\Rightarrow \int_C \frac{1}{z^2}\,dz = 0$$

(2). Direct Calculation: $z(t) = e^{it}$ $0 \le t \le 2\pi$.

$$\int_C \frac{1}{z^2}\,dz = \int_0^{2\pi} e^{-2it} \cdot ie^{it}\,dt = \int_0^{2\pi} i \cdot e^{-it}\,dt = -e^{-it}\Big|_0^{2\pi} = 0$$

$\square$

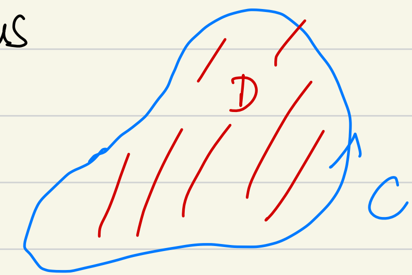
4) Complex integral over closed path:

Recall: Suppose $\vec{F} = \langle M, N \rangle$ is continuously differentiable

(Green's Theorem)

i.e., M_x, M_y, N_x, N_y continuous

then $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F} \, dA$.



Equivalently, $\oint_C M dx + N dy = \iint_D N_x - M_y \, dA$

In particular. If $\operatorname{curl} \vec{F} = 0$, then $\oint_C \vec{F} \cdot d\vec{r} = 0$ $\Leftrightarrow$ path indep $\Leftrightarrow$ ...

Thm: (Cauchy-Goursat Theorem Version 1)

If f is holomophic at all points $\underbrace{\text{interior to}}_{= D}$ and $\underbrace{\text{on}}_{= C}$ a simple closed path C.

then $\boxed{\int_C f(z)\,dz = 0}$.

Pf: Suppose. $f(z) = u(x,y) + i\,v(x,y)$, $\quad C: z(t).\ a \le t \le b$

$$\int_C f\,dz \qquad \int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

$$= \int_C (u+iv)(dx+idy) \qquad = \int_a^b \big(u(x(t),y(t)) + i\,V(x(t),y(t))\big)\cdot\big(x'(t)+i\,y'(t)\big)\,dt$$

$$= \int_C (u\,dx - v\,dy) + i \int_C (u\,dy + v\,dx) \qquad = \int_a^b \underbrace{(u\cdot x' - v y')}_{Re} dt + i \int_a^b \underbrace{(v x' + u y')}_{Im} dt$$

Green thm:

$$= \iint_R \underbrace{(-V_x - u_y)}_{\overset{\shortparallel}{0}} dA \quad \text{Cauchy-Riemann}$$

$$+\, i \iint_R \underbrace{(u_x - V_y)}_{=0} dA$$

$$= 0$$

Re: Let $\vec{F} = \langle u, -v \rangle$.

then $\int_C \vec{F}\cdot d\vec{r} = \int_a^b \langle u, -v\rangle \cdot \langle x', y'\rangle dt = \int_a^b \underbrace{(u x' - v y')}_{Re} dt$

$$\overset{\text{Green}}{=} \iint_D \operatorname{curl} \vec{F}\, dA = \iint_D \underbrace{(-V_x - u_y)}_{\overset{\shortparallel}{0}} dA = 0,$$

Cauchy Riemann

Im: Let $\vec{G} = \langle v, u \rangle$

then $\int_C \vec{G}\cdot d\vec{r} = \int_a^b \langle v, u\rangle \langle x', y'\rangle dt = \int_a^b \underbrace{(v x' + u y')}_{Im} dt$

$$\overset{\text{Green}}{=} \iint_D \operatorname{curl} \vec{G}\, dA = \iint_D \underbrace{(u_x - V_y)}_{\overset{\shortparallel}{0}} dA = 0.$$

Cauchy Riemann

$\square$

Note: Green's thm $\int_C M\,dx + N\,dy = \iint (N_x - M_y)\,dA$

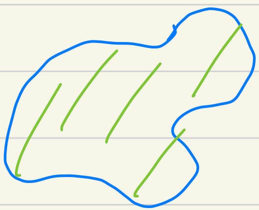
requires M, N have continuous partial derivatives.

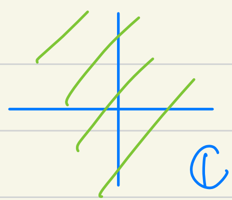
Thus, U_x, U_y, V_x, V_y are cont. $\Leftrightarrow$ $f'(z)$ continuous.

Goursat: The assumption "f' continuous" can be dropped!

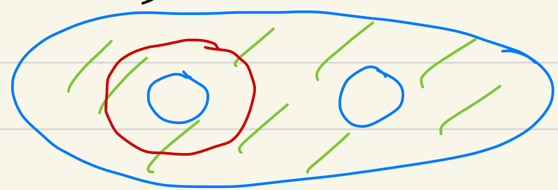
Def: A region $D \subset \mathbb{C}$ is called simply-connected if the interior of any simple closed curve in D is also contained in D.

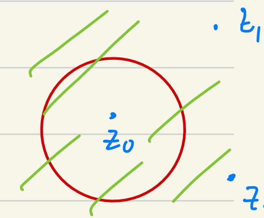
"no holes"





Simply-connected





NOT Simply-connected

$\mathbb{C} \setminus \{z_0\}$ ⋮

Cauchy-Goursat Thm : If f is holomorphic in a simply-connected region D.
(General Version)

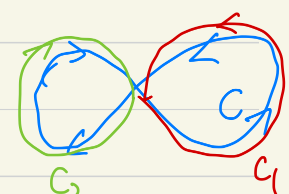
then $\int_C f(z)\,dz = 0$ $\forall$ closed curves C in D.

$\Leftrightarrow$ f is path-indep in D

$\Leftrightarrow$ f has an anti-derivative in D. $f(z) = F'(z)$.

pf: Apply Version 1 of Cauchy-Goursat Thm $\Rightarrow$ $\int_C f(z)dz = 0$ $\forall$ simple closed curve

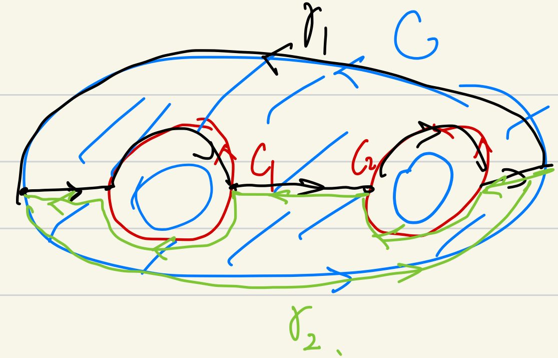
For general closed curve, break into a sum of simple closed curve



$$\int_C f\,dz = \int_{C_1} f\,dz + \int_{C_2} f\,dz = 0$$

$\square$.

Multiply-Connected Region?



Thm. If f is holomorphic in the domain inside C and exterior to C_k,

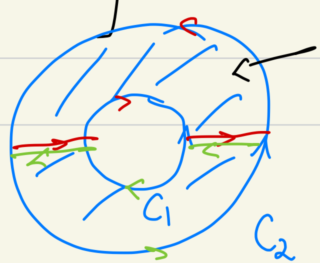
then $\boxed{\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz.}$

$$\gamma_1 \cup \gamma_2 = C \cup -C_1 \cup -C_2$$

Convention: All closed curves are counter-clockwise oriented.

Pf: Decompose D into smaller simply connected regions. then apply Cauchy-Goursat Thm $\square$

Special Case:



f holomorphic

$$\Rightarrow \boxed{\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.}$$

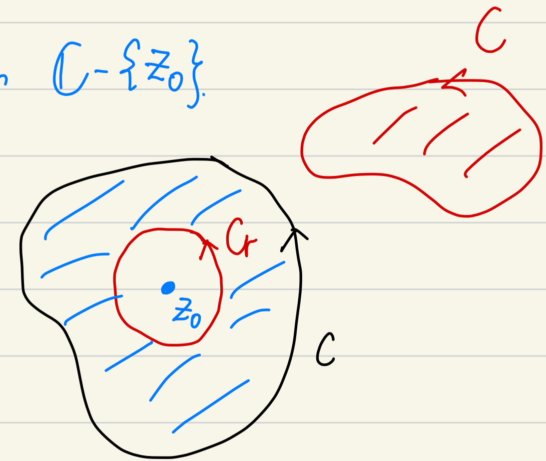
Ex: Evaluate $\int_C \frac{1}{z-z_0}\,dz$ for simple closed curve C in the plane.

Sol: Note that $\frac{1}{z-z_0}$ is defined and holomorphic on $\mathbb{C}-\{z_0\}$.

Two cases:

(1) C not around z_0

Cauchy Thm $\Rightarrow$ $\int_C \frac{1}{z-z_0}\,dz = 0$ holomorphic



(2) C around z_0. Let C_r small circle inside C of radius r.

$$z(t) = r\cdot e^{it} + z_0 \quad , \quad 0 \le t \le 2\pi$$

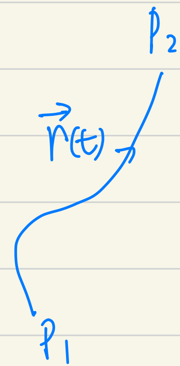
$$\Rightarrow \int_C \frac{1}{z-z_0}\,dz = \int_{C_r} \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{1}{\cancel{r\cdot e^{it}}} \cdot \cancel{r\cdot e^{it}} \cdot i\,dt = 2\pi i.$$

$\square$

Summary:

real line integral

$$\int_\gamma \vec{F} \cdot d\vec{r}$$



$$\vec{F} = \langle M(x,y), N(x,y) \rangle$$

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

$$\int_\gamma \vec{F} \cdot d\vec{r} := \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

dot product

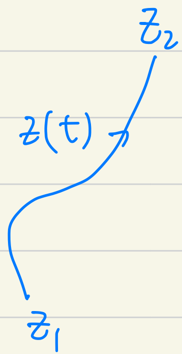
Fundamental Thm of Calculus:

$$\int_\gamma \vec{F} \cdot d\vec{r} = f(P_2) - f(P_1)$$

where $\nabla f = \vec{F}$.

Complex line integral

$$\int_\gamma f(z)\, dz$$



$$f(z) = u(x,y) + i \cdot v(x,y)$$

$$z(t) = x(t) + i\, y(t)$$

$$\int_\gamma f(z)\, dz := \int_a^b f(z(t)) \cdot z'(t)\, dt$$

complex product

$$\int_\gamma f(z)\, dz = F(z_2) - F(z_1)$$

where $F'(z) = f(z)$

Path Indep: $\vec{F} = \nabla f$

$\Leftrightarrow \oint_C \vec{F} \cdot d\vec{r} = 0 \quad \forall$ closed C

$\Leftrightarrow \int_\gamma \vec{F} \cdot d\vec{r}$ path-indep.

$\Updownarrow$

$\operatorname{Curl} \vec{F} = 0$

On simply-connected domain:

$f(z) = F'(z)$

$\Leftrightarrow \int_C f(z)\,dz = 0 \quad \forall$ closed C

$\Leftrightarrow \int_\gamma f(z)\,dz$ path indep.

$\Updownarrow$

f is holomorphic.