Note 2

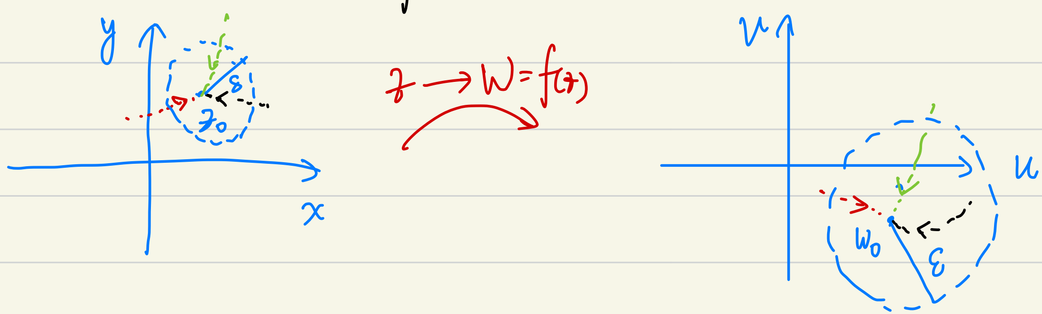
§ Holomorphic / Analytic Functions.

Roughly speaking, $f(z)$ is holomorphic/analytic if it has a complex derivative $f'(z)$.

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}.$$

Understand this limit!

Def (Limits): $\lim_{z \to z_0} f(z) = w_0$ if for all $\varepsilon > 0$, there is $\delta > 0$ s.t. $|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.



- Limit of Complex f^n can be reformulated in terms of real f^n.

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

Suppose $z_0 = x_0 + iy_0$, $\quad w_0 = u_0 + i\,v_0$

Then $\displaystyle\lim_{z \to z_0} f(z) = w_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$

and $\displaystyle\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

Ex: • "Obvious Limit": $\lim\limits_{z\to 2} \dfrac{z^2+2}{z^3+1} = \dfrac{6}{9}$

• "Two-path Test": $\lim\limits_{z\to 0} \dfrac{z}{\bar{z}}$:

Along real axis $z = x+0i$, $\dfrac{z}{\bar{z}} = \dfrac{x}{x} = 1 \Rightarrow \lim = 1$

$\neq$

Along imaginary axis $z = 0+yi$, $\dfrac{z}{\bar{z}} = \dfrac{yi}{-yi} = -1 \Rightarrow \lim = -1$

$\Rightarrow$ limit does not exist.

- Limit involving ∞ : Extended Complex plane $= \mathbb{C} \cup \{\infty\}$
  $\quad\quad$ ||
  add one point at infinity

Key idea : " $\frac{1}{\infty} = 0$ "

- $\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$
- $\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$
- $\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$

Ex: $\lim_{z\to\infty} \frac{2z+i}{z+1}$.

Intuitively: $\lim = \frac{2\infty}{\infty} = 2$.

$= \lim_{z\to 0} \frac{\frac{2}{z}+i}{\frac{1}{z}+1}$

$= \lim_{z\to 0} \frac{2+iz}{1+z}$

$= 2$.

Def: · f Continuous at z_0 if $\lim\limits_{z\to z_0} f(z)$ exists and equals $f(z_0)$.

· f Continuous on a region D if it's continuous at every pt $z \in D$

Likewise, $f(z) = f(x+iy) = u(x,y) + i \cdot v(x,y)$ Continuous
$\Longleftrightarrow$ $u(x,y)$ and $v(x,y)$ are continuous.

Suppose $f(z)$ and $g(z)$ continuous on D. Then

- $f(z) \pm g(z)$ cont on D
- $f(z)\, g(z)$ cont on D
- $f(z) / g(z)$ cont. on D except (possibly) at pt where $g(z) = 0$.
- If h cont. on $f(D)$ then $h(f(z))$ cont. on D.

Ex.

- $e^z = e^{x+iy} = \underbrace{e^x \cos y}_{u(x,y)} + i \cdot \underbrace{e^x \cdot \sin y}_{v(x,y)}$ Cont.
- e^{z^2} Cont.
- $\cos z = (e^{iz} + e^{-iz})/2$
- $P(z)$, $Q(z)$ polynomials, $\underbrace{\frac{P(z)}{Q(z)}}_{\text{rational function}}$ Cont. except at roots of $Q(z)$.

§ Derivatives

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$ derivative of f at z_0

If the limit exists, f is (complex) differentiable / holomorphic at z_0.

Ex: · $f(z) = z^2$. $f'(z_0) = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} z + z_0 = 2z_0$

$\Rightarrow f'(z_0) = 2z_0$

No limit

· $f(z) = \bar{z}$. $f'(z_0) = \lim_{z \to z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} = \lim_{w \to 0} \frac{\bar{w}}{w}$. where $w = z - z_0$.

$\Rightarrow f$ No-where differentiable.

Derivative Rules:

- Sum rule: $\frac{d}{dz}(f(z)+g(z)) = f' + g'$
- Product rule: $\frac{d}{dz}(f(z)\cdot g(z)) = f'g + fg'$
- Quotient rule: $\frac{d}{dz}(f(z)/g(z)) = \frac{f'g - fg'}{g^2}$
- Chain rule: $\frac{d}{dz} g(f(z)) = g'(f(z)) \cdot f'(z)$
- Inverse rule: $\frac{d}{dz} f^{-1}(z) = \frac{1}{f'(f^{-1}(z))}$

Same as real-valued functions.

Pf: ("replace" z by x in MATH 1010)

$$\frac{d}{dz}(f(z)\cdot g(z)) = \lim_{z\to z_0} \frac{f(z)g(z) - f(z_0)\cdot g(z_0)}{z - z_0}$$

$$= \lim_{z\to z_0} \frac{f(z)g(z) - f(z_0)g(z) + f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0}$$

$$= \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}\cdot g(z) + \lim_{z\to z_0} f(z_0)\cdot \frac{g(z) - g(z_0)}{z - z_0}$$

$$= f'(z_0)\cdot g(z_0) + f(z_0)\cdot g'(z_0).$$

$\square$

§ Cauchy-Riemann Equations.

Necessary Condition

"Two path Test"

Thm: If $f(z) = u(x,y) + i \cdot v(x,y)$ is differentiable at $z_0 = x_0 + iy_0$ then partial derivatives of u and v exist at (x_0, y_0) and satisfy the Cauchy Riemann equations: $\boxed{u_x = v_y \quad , \quad u_y = -v_x}$ at (x_0, y_0)

Moreover, $f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)$ ← horizontal path

$= v_y(x_0, y_0) - i\, u_y(x_0, y_0)$ ← vertical path.

pf: If $f'(z_0) = \lim\limits_{\Delta z \to 0} \dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ exists

it's the same either approching z_0 horizontally or vertically.

(1). Horizontal approach: $\Delta y = 0$. $\Delta z = \Delta x$

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x+iy_0) - f(x_0+iy_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(u(x_0+\Delta x, y_0) + i\,v(x_0+\Delta x, y_0)) - (u(x_0,y_0) + i\,v(x_0,y_0))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + \lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x} \cdot i$$

$$= u_x(x_0,y_0) + i \cdot v_x(x_0,y_0)$$

(2) Vertical approach: $\Delta x = 0$. $\Delta z = i \cdot \Delta y$

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{\left(u(x_0, y_0+\Delta y) + i v(x_0, y_0+\Delta y)\right) - \left(u(x_0, y_0) + i v(x_0, y_0)\right)}{i \cdot \Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i \cdot \Delta y} + \lim_{\Delta y \to 0} \frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{i \cdot \Delta y} \cdot i$$

$$= v_y(x_0, y_0) - i\, u_y(x_0, y_0)$$

Comparing (1) and (2) proves the theorem. $\square$

Ex: • $f(z) = z^2 = (x+iy)^2 = \underbrace{x^2-y^2}_{u} + \underbrace{2xy}_{v} \cdot i$

$$\begin{array}{ll} u_x = 2x & v_x = 2y \\ u_y = -2y & v_y = 2x \end{array}$$

CR equations hold

• $f(z) = \bar{z} = x - iy$

$$\begin{array}{ll} u_x = 1 & v_x = 0 \\ u_y = 0 & v_y = -1. \end{array}$$

$u_x \neq v_y$. $\Rightarrow$ $f(z)$ is nowhere differentiable

• $f(z) = |z|^2 = z \cdot \bar{z} = \overset{u}{x^2+y^2} \qquad \overset{v}{\overset{\shortparallel}{0}}$

$$\begin{array}{ll} u_x = 2x & v_x = 0 \\ u_y = 2y & v_y = 0. \end{array}$$

CR equations $\Rightarrow$ f is not differentiable except (possibly) at $z_0 = 0$

CR equations as Sufficient Condition:

Thm: Suppose $f(z) = u(x,y) + i\,v(x,y)$

If u, v satisfies Cauchy–Riemann equations

AND all partial derivatives u_x, u_y, v_x, v_y are continuous at (x_0, y_0).

Then f is differentiable at $z_0 = x_0 + i \cdot y_0$, and

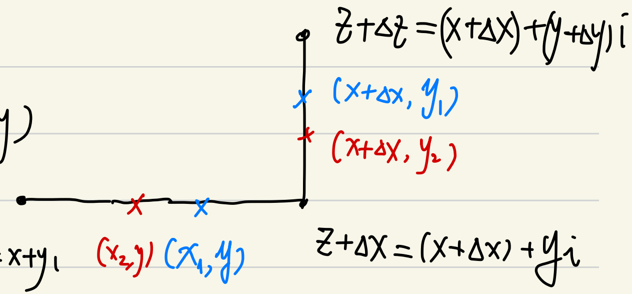
$$f'(z_0) = u_x(x_0, y_0) + i\,v_x(x_0, y_0) = v_y(x_0, y_0) - i \cdot u_y(x_0, y_0).$$

Note: u, v differentiable $\Rightarrow$ $\begin{matrix} u_x & v_x \\ u_y & v_y \end{matrix}$ exists + continuous

$\Leftarrow$

Pf:

$$f(z+\Delta z)-f(z) = u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y) - u(x,y) - i\,v(x,y)$$



real part $\qquad\qquad$ imaginary part

$$= \begin{pmatrix} u(x+\Delta x, y+\Delta y) - u(x+\Delta x, y) \\ + u(x+\Delta x, y) - u(x,y) \end{pmatrix} + \begin{pmatrix} v(x+\Delta x, y+\Delta y) - v(x+\Delta x, y) \\ + v(x+\Delta x, y) - v(x,y) \end{pmatrix}\cdot i$$

(Mean-Value Thm)

$$= \begin{pmatrix} \Delta y\cdot u_y(x+\Delta x, y_1) \\ + \\ \Delta x\cdot u_x(x_1, y) \end{pmatrix} + \begin{pmatrix} \Delta y\cdot v_y(x+\Delta x, y_2) \\ + \\ \Delta x\cdot v_x(x_2, y) \end{pmatrix}\cdot i \quad \text{for } \begin{array}{l} x \le x_1, x_2 \le x+\Delta x \\ y \le y_1, y_2 \le y+\Delta y \end{array}$$

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta x\cdot\big(u_x(x,y)+v_x(x,y)\,i\big) + \Delta y\cdot\big(\underbrace{u_y(x,y)}_{=-v_x} + \underbrace{v_y(x,y)}_{=u_x}\,i\big)}{\Delta x + i\,\Delta y} = u_x + v_x\cdot i$$

$\square$

Application:

- Exponential Function $e^{z} = e^{x+iy} = \underbrace{e^{x}\cos y}_{u} + i\cdot \underbrace{e^{x}\sin y}_{v}$

$$u_x = e^{x}\cos y \qquad v_x = e^{x}\sin y$$
$$u_y = -e^{x}\sin y \qquad v_y = e^{x}\cos y$$

$\Rightarrow e^{z}$ differentiable and $\frac{d}{dz}e^{z} = e^{x}\cos y + i\, e^{x}\cdot \sin y$

$$= e^{z}.$$

- Functions not expressed in terms of z.

Ex: $f(z) = \underbrace{\sin x \cosh y}_{u} + i\,\underbrace{\cos x \cdot \sinh y}_{v}$

(Can write in terms of z and $\bar{z}$. but $\bar{z}$ is not diff.)

$U_x = \cos x \cdot \cosh y \qquad V_x = -\sin x \sinh y$

$U_y = \sin x \sinh y \qquad V_y = \cos x \cosh y$

Recall: $\sinh y = \dfrac{e^y - e^{-y}}{2} \qquad \sinh' = \cosh$

$\cosh x = \dfrac{e^y + e^{-y}}{2} \qquad \cosh' = \sinh$

Hence, f is differentiable

$f'(z) = \cos x \cosh y - i \sin x \sinh y$

- $f'(z) = 0$ on a disk $\Longleftrightarrow$ $f(z)$ is constant

pf: $f'(z) = 0$ $\Rightarrow$ $u_x = u_y = v_x = v_y = 0$

Cauchy Riemann eq.

Multi-variable Calculus $\Rightarrow$ u, v constant.

$\Rightarrow$ $f = u + iv$ constant.

$\square$

Def: A function that is differentiable at every pt in the entire complex plane $\mathbb{C}$ is called an **entire function**.

Examples:

- $f(z) = z^n$. $n \geq 0$ integer. <u>entire</u>
  $f'(z) = n \cdot z^{n-1}$
- $P(z)$ polynomial $=$ sum of monomials. <u>entire</u>
- $f(z) = \frac{1}{z^n}$ $n \geq 0$ integer. <u>not</u> entire. Domain of Def $= \mathbb{C} - \{0\}$
  $f'(z) = (-n) \frac{1}{z^{n+1}}$ $\forall z \neq 0$

- $\frac{P(z)}{Q(z)}$ rational function. Assume P, Q have no common roots
  then Domain of Def $= \mathbb{C} - \{\text{roots of } Q\}$

  $$f'(z) = \frac{PQ' - P'Q}{Q^2}$$ (quotient rule)

- $f(z) = e^z$ exponential <u>entire</u>

  $$f'(z) = e^z.$$

- $$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$ <u>entire</u>

  $$\sin' z = \cos z \qquad \cos' z = -\sin z$$

- $$\sinh(z) = \frac{e^z - e^{-z}}{2} \qquad \cosh(z) = \frac{e^z + e^{-z}}{2}$$ <u>entire</u>

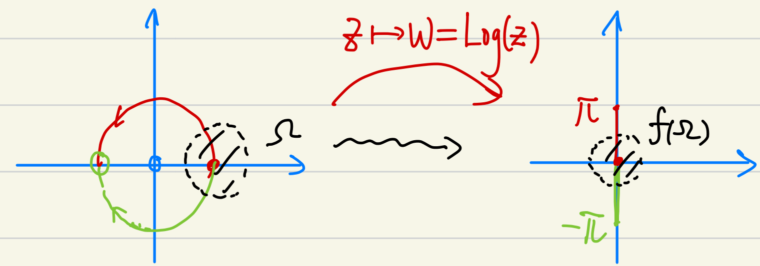
  $$\sinh'(z) = \cosh(z) \qquad \cosh'(z) = \sinh(z).$$

- $\log z = \ln|z| + i\arg(z)$ multi-valued

$\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}(z)$ Domain of def $= \mathbb{C} - \{0\}$

(Even) not continuous!

Recall $-\pi < \mathrm{Arg} \le \pi$



Can solve the problem by "shrinking" the domain (branch cut)

$$\frac{d}{dz}\log z = \frac{1}{z}.$$

- $z^c \quad c \in \mathbb{C}$. $\qquad z^c = e^{c \cdot \log(z)}$. Same problem

After branch cut, $\quad \frac{d}{dz} z^c = \frac{d}{dz} e^{c \cdot \log z} = e^{c \log z} \cdot c \cdot \frac{1}{z} = c \cdot z^{c-1}$.

- $\sin^{-1} z \qquad$ Recall for real-value function

$$y = \sin^{-1} x \qquad x, y \in \mathbb{R}$$

$$\Rightarrow \quad x = \sin y$$

$$\Rightarrow \quad dx = \cos y \cdot dy$$

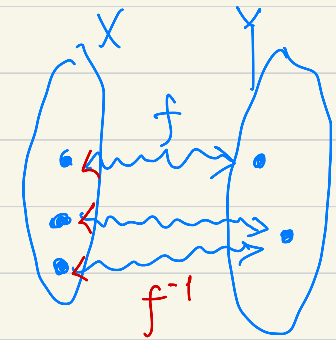
$$\Rightarrow \quad (\sin^{-1} x)' = \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

In the Complex case, $\boxed{\sin^{-1} z = -i \log (iz + (1-z^2)^{1/2})}$

$$\Rightarrow \quad \frac{d}{dz} \sin^{-1} z = -i \frac{1}{iz + (1-z^2)^{1/2}} \cdot \left(i + \frac{1}{\cancel{2}}(1-z^2)^{-1/2} \cdot (-\cancel{2}z)\right) = (1-z^2)^{-\frac{1}{2}}.$$

Appendix I Branch and Branch Cut

- A function in the classical sense is single-valued. $y=f(x)$

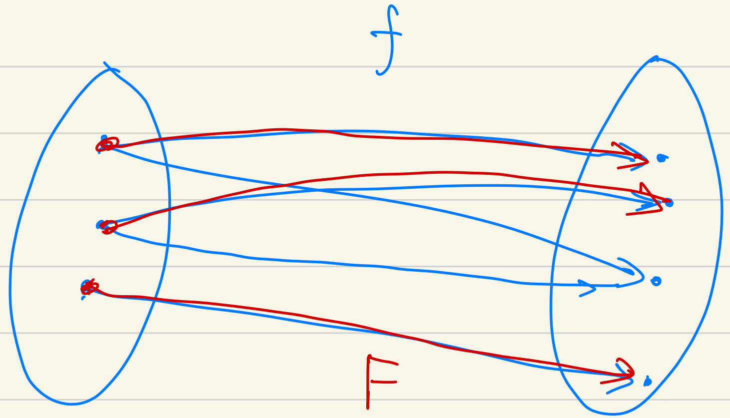


- Yet, a multi-valued "function" arise naturally.
  - Inverse f^{-1} of a function that is not injective.

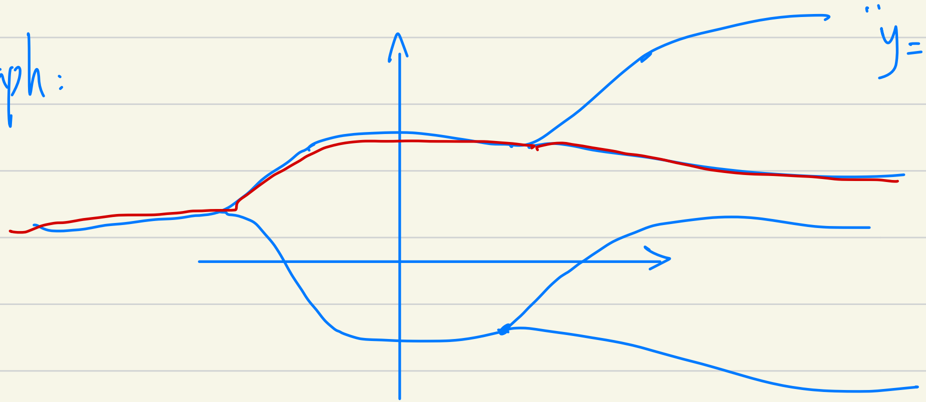
    e.g. inverse of x^2. i.e., $\sqrt{x}$.

- A branch of a multi-valued function f is a single-value function F

  s.t. $F(x)$ is one of the values of $f(x)$.



Graph:



"$y = f(x)$"

$y = F(x)$ a branch

- $x = \sqrt{y} \geq 0$ is secretely a branch of the inverse f^n of $y = x^2$
- $\arg : \mathbb{C} - \{0\} \to \mathbb{R}$ is a multi-valued function.

By choosing $-\pi < \arg(z) \leq \pi$. we obtain a principal branch $\operatorname{Arg}(z)$.

In general, $\forall \theta \in \mathbb{R}$, if we require $\theta < \arg(z) \leq \theta + 2\pi$
or $\theta \leq \arg(z) < \theta + 2\pi$

then we can obtain another branch of $\arg(z)$.

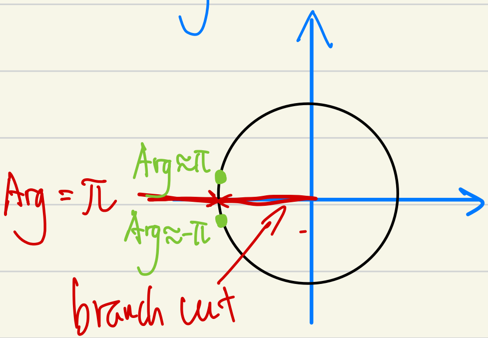
- Correspondingly, each branch of $\arg(z)$ gives a branch of the log function

$$\log(z) = \ln|z| + i \arg(z)$$

Often, a branch F is not continuous / differentiable everywhere in its domain of definition.

Branch Cut : "Cut" points out of domain to remove pts of discontinuity / non-differentiability.

Ex. (1) $\mathrm{Arg}: \mathbb{C}-\{0\} \longrightarrow \mathbb{R}$

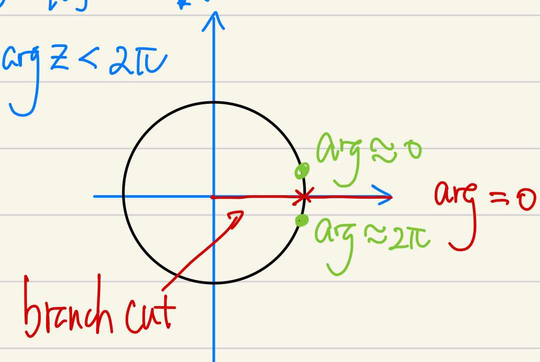


Arg continuous on $\mathbb{C}-\{\text{negative real axis}\}$

$= \{ z \in \mathbb{C} \mid -\pi < \mathrm{Arg}\, z < \pi \}$

(2) $\arg.\ \mathbb{C}-\{0\} \longrightarrow \mathbb{R}$

branch $0 \le \arg z < 2\pi$



Continuous on $\mathbb{C} - \{\text{positive real axis}\}$

$= \{ z \in \mathbb{C} \mid 0 < \arg z < 2\pi \}$

Appendix II Differentiable v.s Analytic v.s. Holomorphic

- For real function $y = f(x)$, $x, y \in \mathbb{R}$

f differentiable $=$ f' exists
f twice-differentiable $=$ f' and f'' exist.

f k-time differentiable $=$ $f', f'', \ldots, f^{(k)}$ exist
f smooth / ∞ differentiable $=$ $f^{(k)}$ exist $\forall k$.

f analytic $=$ f smooth and $f =$ Taylor Series

i.e., $f(x) = f(x_0) + f'(x_0)\cdot(x - x_0) + \frac{f''(x_0)}{2}\cdot(x-x_0)^2 + \ldots$ near x_0.

Ex: $f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$.

- f differentiable: $f'(x) = \begin{cases} e^{-\frac{1}{x}} \cdot \frac{1}{x^2} & x > 0. \\ 0 & x \leq 0 \end{cases}$

  Note: $f'(0) = \lim\limits_{x \to 0^+} \dfrac{e^{-\frac{1}{x}}}{x} = \lim\limits_{y \to \infty} y \cdot e^{-y} = \lim\limits_{y \to \infty} \dfrac{y}{e^y} = 0$

- f smooth: Induction: $f^{(k)}(x) = \begin{cases} \text{formula} & x > 0 \\ 0 & x \leq 0. \end{cases}$

- In particular, Taylor Series near $x=0$: $f(0) + f'(0) \cdot x + \frac{f''(0)}{2} \cdot x^2 + \cdots = 0$

$\Rightarrow$ f not analytic at 0.

- For complex function $w = f(z)$ $z, w \in \mathbb{C}$.

Can still talk about complex differentiable, twice differentiable, ⋯ analytic.

Amazingly, as we'll see in MATH2230:

f is differentiable $\Leftrightarrow$ f is ∞-differentiable $\Leftrightarrow$ f is analytic!

To avoid confusion, we'll use the terms: holomorphic := complex differentiable.

In a month, we'll prove holomorphic = analytic.