Note 7

§ Application I: Real Integral

- Definite v.s. Indefinite Integral (Anti-derivative)

$$\int_a^b f(x)\,dx \in \mathbb{R} \qquad\qquad \int f(x)\,dx = F(x)$$

$$= F(b) - F(a)$$

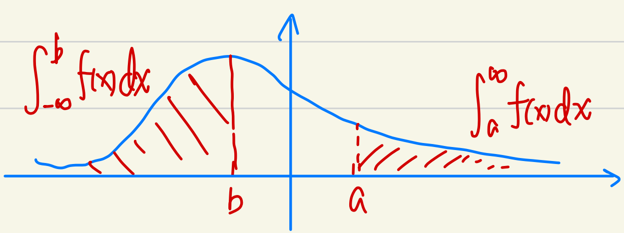
Fund. Thm of Calculus

Question: Evaluate definite integral without anti-derivative? (e.g. $\int_0^{2\pi} \frac{1}{1+a\sin\theta}\,d\theta$)

Improper Integral. • "One-side infinity":

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx := \lim_{a\to-\infty} \int_a^b f(x)\,dx$$



• "Two-side infinity": $$\boxed{\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &:= \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to\infty}\int_0^b f(x)\,dx \\ &= \int_{-\infty}^0 f(x)\,dx + \int_0^{\infty} f(x)\,dx\end{aligned}} \qquad (1)$$

Cauchy Principal value: $$\boxed{P.V. \int_{-\infty}^{\infty} f(x)\,dx := \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx} \qquad (2)$$

Question: Limit exist?

Note if (1) exists, then $$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = \lim_{R\to\infty}\left[\int_{-R}^{0} f(x)\,dx + \int_0^R f(x)\,dx\right]$$
$$= \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$$

$$\Rightarrow P.V. \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

On the other hand, $\exists f(x)$ s.t. P.V. $\int_{-\infty}^{\infty} f(x)\,dx$ exists but $\int_{-\infty}^{\infty} f(x)\,dx$ not.

Ex: $\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$ does not exist

$$\text{P.V.} \int_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty} \int_{-R}^{R} x\,dx = \lim_{R\to\infty} \cancel{\left[\frac{x^2}{2}\right]_{-R}^{R}} = 0$$

0.

- Hence, when we compute improper integral from $-\infty$ to ∞,
  we typically refer to P.V. $\int_{-\infty}^{\infty} f(x)\,dx$. Cauchy Principal value.

- Another common situation: $f(x)$ even function. then $\int_{0}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx$ (if exists)

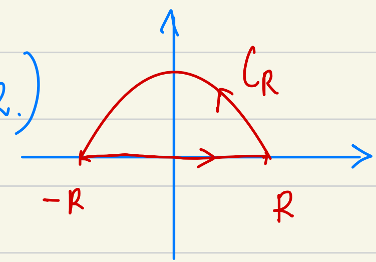
$$\Rightarrow \int_{0}^{\infty} f(x)\,dx = \frac{1}{2}\cdot \text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx$$

General stretegy of computing $P.V.\int_{-\infty}^{\infty} f(x)\,dx$

(1) Find a complex function $g(z)$ related to $f(x)$ (e.g. $g(x)=f(x)\ \forall x\in\mathbb{R}$)

(2) Take a closed curve C that includes the segment $[-R,R]$ in the real axis.

(e.g. $C=[-R,R]\cup C_R$, where C_R top half of circle $|z|=R$.)



(3). Prove $\lim_{R\to\infty}\int_{C_R} g(z)\,dz = 0$.

(4) Evaluate $\int_C f(z)\,dz$, by residue thm, etc

$$\lim_{R\to\infty}\int_{-R}^{R} g(z)\,dz = P.V.\int_{-\infty}^{\infty} f(x)\,dx.$$

Ex 1: $\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx$

Ex 2: $\int_0^{\infty} \frac{1}{(1+x^2)^2}\,dx$

Ex 3: $\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos x}{x^2+b^2}\,dx$ $\quad b>0.$ (trigonometric integral)

Ex 4: $\text{P.V.} \int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2+3}\,dx$ (Jordan's Lemma)

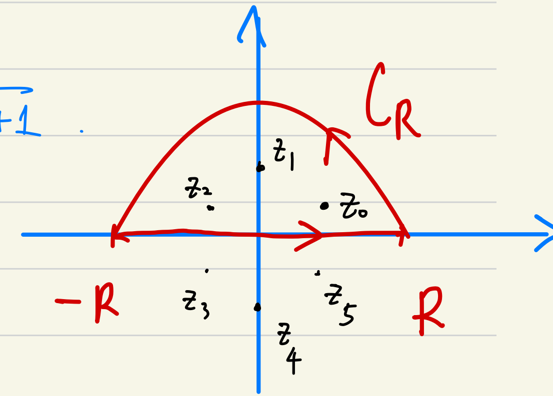
Ex 5: $\int_0^{\infty} \frac{\sin x}{x}\,dx$ (Dirichlet integral. singularities)

Ex 6: $\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$ $\quad 0<a<1$ (Rectangular countour)

Ex 7: $\int_0^{2\pi} \frac{1}{1+a\sin\theta}\,d\theta$ $\quad -1<a<1,\ a\neq 0.$ (proper definite integral)

Example 1: p.v. $\int_{-\infty}^{\infty} \frac{1}{x^6+1}\, dx$.

Naturally, consider the complex function $g(z) = \frac{1}{z^6+1}$.



- Evaluate $\int_{C_R} g(z)\, dz = \int_{C_R} \frac{1}{z^6+1}\, dz$.

$$\left| \int_{C_R} \frac{1}{z^6+1}\, dz \right| \le \max_{z \in C_R} \left| \frac{1}{z^6+1} \right| \cdot (\text{length of } C_R)$$

$$= \frac{1}{R^6-1} \cdot \pi R \longrightarrow 0 \quad \text{as } R \to \infty$$

$$\Rightarrow \lim_{R\to\infty} \int_{C_R} g(z)\, dz = 0 .$$

- Evaluate $\int_{C} g(z)\, dz = 2\pi i \sum \text{Res}$

$C = C_R \cup [-R, R]$

Note $\frac{1}{z^6+1} = \prod_{k=1}^{6} \frac{1}{(z-z_k)}$ where $z_k = e^{i\frac{\pi+2k\pi}{6}}$.

$\left(z^6 = -1 = e^{i\cdot\pi} \Rightarrow z_k = e^{i\left(\frac{\pi+2k\pi}{6}\right)} \quad k=0,1,\cdots,5 \right.$

The curve C enclose 3 singularities z_0, z_1, z_2

$$\Rightarrow \int_C g(z)\,dz = 2\pi i \sum_{k=0}^{2} \operatorname{Res}(g, z_k) = 2\pi i\cdot\left(-\frac{1}{6}\right)\cdot\underbrace{\left(e^{i\frac{\pi}{6}} + e^{i\frac{\pi}{2}} + e^{i\frac{5\pi}{6}}\right)}_{2i}$$

$$= \frac{2}{3}\pi$$

$$\left(\text{Compute } \operatorname{Res}(g, z_k) = \frac{1}{(z^6+1)'}\Big|_{z=z_k} = \frac{1}{6z_k^5} = -\frac{1}{6}z_k \right)$$

Thus $\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx + \underbrace{\lim_{R\to\infty} \int_{C_R} g(z)\,dz}_{0} = \int_C g(z)\,dz = \frac{2}{3}\pi$

$\| $

P.V. $\int_{-\infty}^{\infty} f(x)\,dx$.

$\square$

- Decay of functions.

Thm: Suppose $f(z)$ is defined in the upper half-plane. If there is $a>1$ and $M>0$

s.t. $\boxed{|f(z)| < \frac{M}{|z|^a}}$ for large z (~ decay faster than $\frac{1}{z}$)

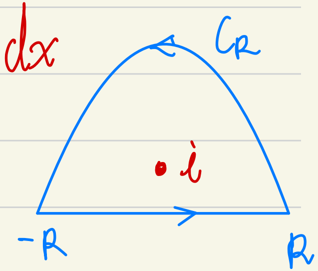
Then $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$. where C_R is the upper half circle $|z|=R$

pf: $$\left|\int_{C_R} f(z)\,dz\right| \le \max_{z\in C_R}|f(z)| \cdot (\text{length of } C_R) < \frac{M}{R^a}\cdot \pi R = \frac{M\pi}{R^{a-1}}$$

Since $a>1$, limit goes to 0 when $R \longrightarrow \infty$ $\square$

Example 2: Compute $\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = \frac{\pi}{4}$ $= \frac{1}{2}\,P.V.\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx = \frac{\pi}{2}$

↑ even function



Let $f(z) = \frac{1}{(1+z^2)^2}$. $f(z) \sim \frac{1}{z^4}$ when z large.

Hence, previous thm applies and $P.V.\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx = \int_C f(z)\,dz = 2\pi i \cdot \sum \mathrm{Res} = \frac{\pi}{2}$.

Compute residue: $f(z) = \frac{1}{(z+i)^2(z-i)^2}$ has singularities at $\pm i$.

Since i is a pole of order 2, then $\mathrm{Res}(f, z_0 = i) = \frac{g'(z_0)}{(2-1)!} = g'(i)$

where $g(z) = (z-z_0)^2 f(z) = \frac{1}{(z+i)^2}$.

$\Rightarrow g'(z) = -2 \cdot \frac{1}{(z+i)^3}$ $\Rightarrow g'(i) = \frac{-2}{8i^3} = -\frac{i}{4}$

§ Trigonometric Integrals.

In Fourier analysis, we often encounter integral of the form

$$\int_{-\infty}^{\infty} f(x)\cdot \sin ax \, dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x) \cos ax \, dx$$ (Fourier coefficient)

W.L.O.G, assume $a>0$.

To do such integral, we may consider $\int_C f(z)\cdot \sin az \, dz$ hard to control the size

Problem: $\sin az = \frac{e^{iaz} - e^{-iaz}}{2i} = \frac{e^{-ay+iax} - e^{ay-iax}}{2i}$

(Ex.) $|\sin az| \sim e^{ay}$ as $y \to +\infty$

Remedy: Consider instead $\int_C f(z)\, e^{i \cdot az}\, dz$

Note that $|e^{i \cdot az}| = |e^{-ay + iax}| = e^{-ay} \leq 1$ on upper half plane.

In favorable case, can prove $\lim\limits_{R\to\infty} \int_{C_R} f(z) \cdot e^{iaz}\, dz = 0$

$$\Rightarrow \quad \int_C f(z) \cdot e^{i \cdot az}\, dz = \lim_{R\to\infty} \int_{-R}^{R} f(x) \cdot e^{iax}\, dx$$

$$= \underbrace{P.V. \int_{-\infty}^{\infty} f(x) \cdot \cos ax\, dx}_{\text{Real}} + i \cdot \underbrace{P.V. \int_{-\infty}^{\infty} f(x) \cdot \sin ax\, dx}_{\text{Imaginary}}$$

Compute this by residue thm.

Example 3: Suppose $b > 0$. Compute $\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + b^2} \, dx$

Let $g(z) = \frac{e^{iz}}{z^2 + b^2}$

For $z = x + iy$ with $y \geq 0$. $|g(z)| = \frac{|e^{i(x+iy)}|}{|z^2 + b^2|} = \frac{e^{-y}}{|z^2 + b^2|}$ ≤ 1 decays $\sim \frac{1}{|z|^2}$.

Hence $\lim_{R \to \infty} \int_{C_R} g(z) \, dz = 0$

Note that $g(z)$ has two poles $\pm bi$, among which bi lies inside C.

$$\text{Res}(g(z), bi) = \frac{e^{-b}}{2bi}$$

$$\Rightarrow \int_C g(z) \, dz = 2\pi i \cdot \text{Res}(g, b \cdot i) = \frac{\pi \cdot e^{-b}}{b} = \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + b^2} \qquad \square$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + b^2} \, dx = 0, \quad \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + b^2} \, dx = \frac{\pi e^{-b}}{b}$$

To motivate Jordan's lemma, look at the following example.

Example 4: p.v. $\int_{-\infty}^{\infty} \frac{x \sin 2x}{x^2+3}\, dx$

By previous discussion, we consider $g(z) = \frac{z}{z^2+3} \cdot e^{i \cdot 2z}$.

Note $\left| \frac{z}{z^2+3} e^{i \cdot 2z} \right| = \left| \frac{z}{z^2+3} \right| \cdot e^{-2y} \sim \frac{1}{|z|}$

It does not decay fast enough to apply the earlier thm.

...

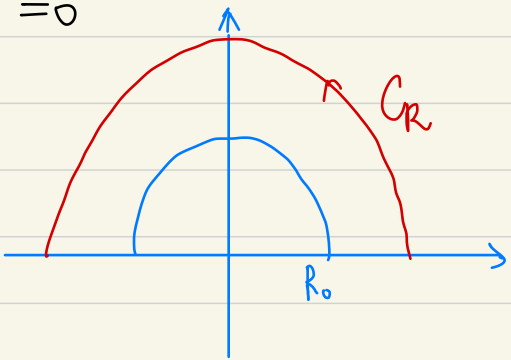
Thm (Jordan's lemma): Suppose

(a). f is analytic on $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0, |z| \geq R_0\}$ for some $R_0 > 0$.

(b). For each $R > R_0$, there's a positive const. M_R s.t. $\boxed{\max_{C_R} |f| \leq M_R}$

and $\boxed{\lim_{R \to \infty} M_R = 0}$.

Then for every $a > 0$, $\lim_{R \to \infty} \int_{C_R} f(z) \cdot e^{iaz} \, dz = 0$



Back to the previous example: $f(z) = \frac{z}{z^2+3}$

(a) Poles at $\pm\sqrt{3}i$, so $f(z)$ is analytic for $R_0 = 2 > \sqrt{3}$

(b) $\max\limits_{z\in C_R} |f(z)| \le M_R = \frac{R}{R^2-3}$ for $R \ge R_0 > \sqrt{3}$

Clearly, $\lim\limits_{R\to\infty} M_R = 0$

Apply Jordan's lemma: $\Rightarrow \lim\limits_{R\to\infty} \int_{C_R} f(z)\cdot e^{i\cdot 2z} dz = 0$.

Hence $\int_C f(z) e^{i\cdot 2z} dz = 2\pi i \operatorname{Res}\left(\frac{z}{z^2+3}\cdot e^{i\cdot 2z}, \sqrt{3}i\right) = 2\pi i\cdot\left(\frac{\sqrt{3}i}{2\sqrt{3}i}\cdot e^{-2\sqrt{3}}\right)$

$= e^{-2\sqrt{3}}\pi i$

So P.V. $\int_{-\infty}^{\infty} \frac{x}{x^2+3}\sin 2x\, dx = e^{-2\sqrt{3}}\cdot\pi$. $\square$

Pf of Jordan's Lemma: For $a>0$, $R>R_0$. $z(\theta)=Re^{i\theta}$

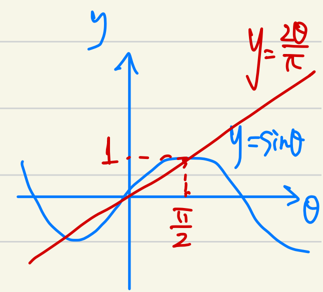
$$\int_{C_R} f(z)\cdot e^{ia\cdot z}\,dz = \int_0^{\pi} f(R\cdot e^{i\theta})\, e^{i\cdot a\cdot R\cdot \overset{=\cos\theta+i\sin\theta}{e^{i\theta}}}\cdot Rie^{i\theta}\,d\theta$$

$$= iR\int_0^{\pi} f(R\cdot e^{i\theta})\, e^{-aR\sin\theta}\cdot e^{i(aR\cos\theta+\theta)}\,d\theta$$

Thus

$$\left|\int_{C_R} f(z)\cdot e^{iaz}\,dz\right| \le R\cdot\int_0^{\pi} |f(R\cdot e^{i\theta})|\cdot e^{-aR\sin\theta}\,d\theta$$

$$\le R\cdot M_R\cdot\int_0^{\pi} e^{-aR\sin\theta}\,d\theta.$$



Key Step: Observe $\sin\theta \ge \frac{2\theta}{\pi}$ for $\theta\in[0,\frac{\pi}{2}]$

$$\text{Then} \quad \int_0^{\pi} e^{-aR\sin\theta}\, d\theta = 2\cdot \int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}\, d\theta$$

$$\left(-aR\sin\theta \leq -\frac{2aR\theta}{\pi}\right) \qquad \leq 2\cdot \int_0^{\frac{\pi}{2}} e^{\frac{-2aR\theta}{\pi}}\, d\theta$$

$$= 2\cdot \left(\frac{\pi}{-2aR}\right)\cdot e^{\frac{-2aR\theta}{\pi}}\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{aR}\cdot\left(1 - e^{-aR}\right)$$

$$\leq \frac{\pi}{aR}$$

$$\text{Therefore,} \quad \left| \int_{C_R} f(z)\cdot e^{iaz}\, dz \right| \leq \cancel{R}\cdot M_R \cdot \frac{\pi}{a\cancel{R}} \longrightarrow 0 \quad \text{as } R \to \infty$$

$\square$

(Example 5)

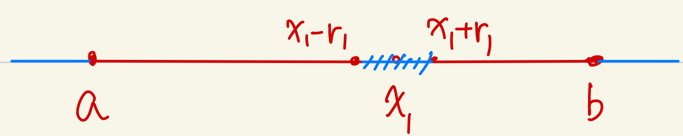
- **Dirichlet's integral**: $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{1}{2}\,\text{P.V.}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$

Consider $\displaystyle\int_C \frac{e^{iz}}{z}\,dz$, which has a pole at $z=0$.

We introduce another type of improper integral.

Def: Suppose $f(x)$ has a (possible) singularity at x_1, then the **Cauchy principal value** is:

$$\boxed{\text{P.V.}\int_a^b f(x)\,dx := \lim \int_a^{x_1-r_1} f(x)\,dx + \int_{x_1+r_1}^{b} f(x)\,dx}$$



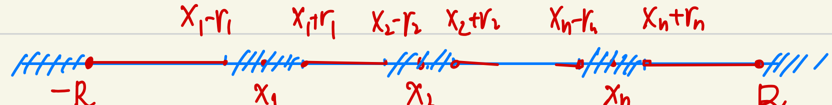
$$\boxed{\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx := \lim_{\substack{R\to\infty \\ r_1\to 0}} \int_{-R}^{x_1-r_1} f(x)\,dx + \int_{x_1+r_1}^{R} f(x)\,dx}$$

Note: If $f(x)$ is continuous (at x_1), then $\displaystyle\lim_{r_1\to 0}\int_{x_1-r_1}^{x_1+r_1} f(x)\,dx = 0$

$$\Rightarrow \quad \text{p.v.}\int_a^b f(x)\,dx = \int_a^b f(x)\,dx \quad \text{"proper" integral.}$$

More generally, if there are multiple points of discontinuity, $x_1 < x_2 < \cdots < x_n$

$$\boxed{\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx := \lim_{\substack{R\to\infty \\ r_1,\cdots,r_n\to 0}} \int_{-R}^{x_1-r_1} f(x)\,dx + \int_{x_1+r_1}^{x_2-r_2} f(x)\,dx + \cdots + \int_{x_n+r_n}^{R} f(x)\,dx}$$

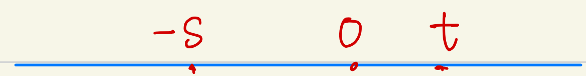


Important: Intervals around each x_i are SYMMETRIC !

Basically, want to "cancel" the contribution near singular points.

Ex: $$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x}\,dx = \lim_{\substack{R\to\infty \\ r\to 0}} \left(\int_{-R}^{-r} \frac{1}{x}\,dx + \int_{r}^{R} \frac{1}{x}\,dx \right) = 0$$

WITHOUT the symmetric assumption:



$$\int_{-R}^{-s} \frac{1}{x}\,dx + \int_{t}^{R} \frac{1}{x}\,dx = \ln(s) - \cancel{\ln(R)} + \cancel{\ln(R)} - \ln(t)$$

does not have a limit when $s, t \to 0$

Back to Dirichlet's Integral $\int_0^\infty \frac{\sin x}{x}\,dx$

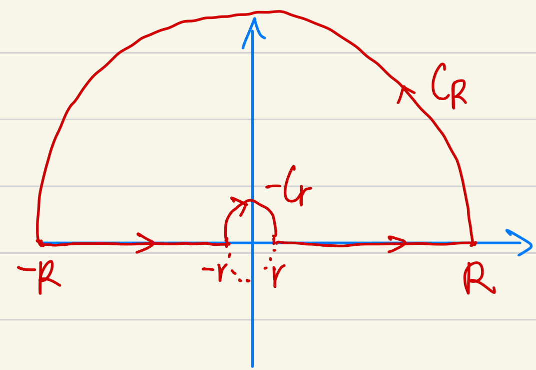
By our earlier method, consider instead the integral of $\int_C \frac{e^{iz}}{z}\,dz$

- $\frac{e^{iz}}{z}$ has no pole inside C. $\Rightarrow \int_C \frac{e^{iz}}{z}\,dz = 0$

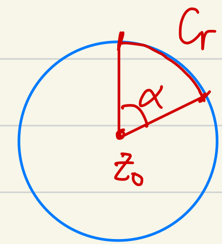
- The function $\frac{1}{z}$ satisfies the condition of Jordan's lemma

$$\Rightarrow \lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z}\,dz = 0$$

- We are left to find out $\lim_{r\to 0} \int_{C_r} \frac{e^{iz}}{z}\,dz$



In general, we study integral over a portion of circle



Thm: Suppose $f(z)$ has a simple pole at z_0.

Let C_r be the circular arc $z(\theta) = z_0 + r \cdot e^{i\theta}$. with $\theta_0 < \theta < \theta_0 + \alpha$.

Then
$$\boxed{\lim_{r \to 0} \int_{C_r} f(z)\,dz = \alpha \cdot i \cdot \operatorname{Res}(f, z_0)}$$

Pf: Take r small enough, s.t. no other singularities in $0 < |z - z_0| < r$.

Laurent Series: $f(z) = \dfrac{b_1}{z - z_0} + \underbrace{a_0 + a_1(z - z_0) + \ldots}_{g(z)}$

$g(z)$ analytic $|z - z_0| < r$

$$\Rightarrow \int_{C_r} f(z)\,dz = \int_{C_r} \frac{b_1}{z-z_0}\,dz + \int_{C_r} g(z)\,dz.$$

Note $\displaystyle \int_{C_r} \frac{b_1}{z-z_0}\,dz = \int_{\theta_0}^{\theta_0+\alpha} \frac{b_1}{r\cdot e^{i\theta}} \cdot ire^{i\theta}\,d\theta = b_1 \cdot i \cdot \alpha = \alpha \cdot i \cdot \mathrm{Res}(f, z_0)$

$$\left|\int_{C_r} g(z)\,dz\right| \le \underbrace{\max|g(z)|}_{=:M} \cdot (\text{length of } C_r) = M\cdot\alpha\cdot r \longrightarrow 0 \text{ when } r \to 0$$

$\square$.

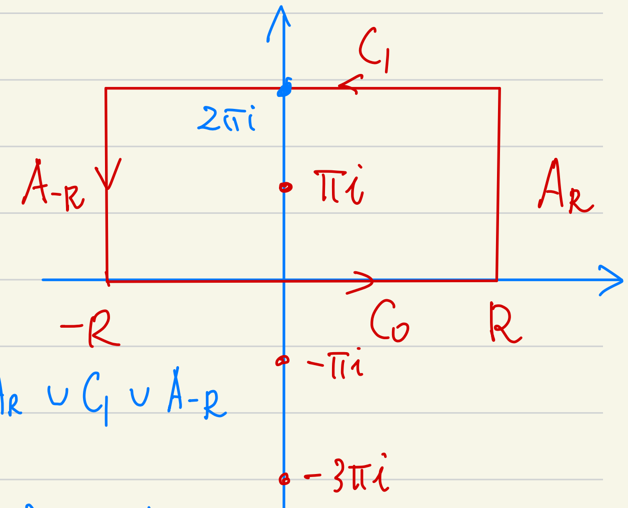
Back to Dirichlet's integral: $\displaystyle \lim_{r\to 0} \int_{C_r} \frac{e^{iz}}{z}\,dz = \pi i \cdot \mathrm{Res}\left(\frac{e^{iz}}{z}, 0\right) = \pi i.$

$$\Rightarrow \lim_{\substack{r\to 0\\ R\to\infty}} \int_{-R}^{-r} \frac{e^{iz}}{z}\,dz + \int_{r}^{R} \frac{e^{iz}}{z}\,dz = \pi i$$

Take the imaginary part $\Rightarrow \displaystyle \int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\,\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \frac{1}{2}\pi$

$\square$

- Rectangle contour C



Example 6: $\quad P.V. \displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx \qquad 0<a<1$

Sol: Consider $\displaystyle\int_C \frac{e^{az}}{1+e^z}\,dz$

where C is a "rectangle" loop $[-R,R] \cup A_R \cup C_1 \cup A_{-R}$

- Note that $f(z) = \dfrac{e^{az}}{1+e^z}$ has poles at $z = \pm\pi i, \pm 3\pi i, \cdots$, among which πi lies inside C.

$$\Rightarrow \int_C \frac{e^{az}}{1+e^z}\,dz = 2\pi i \cdot \operatorname{Res}\Big(\underbrace{\frac{e^{az}}{1+e^z}, \pi i}_{= -e^{-a\pi i}}\Big) = -2\pi i\, e^{a\pi i}.$$

- On C_1, $z = x + 2\pi i$. $\Rightarrow \dfrac{e^{az}}{1+e^z} = \dfrac{e^{ax}\cdot e^{a\cdot 2\pi i}}{1+e^x}$

$$\Rightarrow \int_{C_1} \frac{e^{az}}{1+e^z}\,dz = -e^{a\cdot 2\pi i}\cdot \int_{-R}^{R} \frac{e^{ax}}{1+e^x}\,dx$$

- On A_R, $z = R + yi$.

$$\left| \int_{A_R} \frac{e^{az}}{1+e^z}\,dz \right| \le \max_{z\in A_R} \frac{\overbrace{|e^{az}|}^{=e^{aR}}}{\underbrace{|1+e^z|}_{\ge e^R-1}} \cdot (\overbrace{\text{length of } A_R}^{=2\pi}) \le \frac{e^{aR}}{e^R-1}\cdot 2\pi \longrightarrow 0$$

when $R \to \infty$

$$\Rightarrow \lim_{R\to\infty} \int_{A_R} \frac{e^{az}}{1+e^z}\,dz = 0.$$

- On A_{-R}, $z = -R + yi$

$$\left| \int_{A_{-R}} \frac{e^{az}}{1+e^z}\, dz \right| \le \max_{z \in A_{-R}} \frac{\overbrace{|e^{az}|}^{\le e^{-aR}}}{\underbrace{|1+e^z|}_{\ge 1-e^{-R}}} \cdot \overbrace{(\text{length of } A_{-R})}^{\le 2\pi} \le \frac{e^{-aR}}{1-e^{-R}} \cdot 2\pi$$

$$\Rightarrow \lim_{R\to\infty} \int_{A_{-R}} \frac{e^{az}}{1+e^z}\, dz = 0. \qquad = \frac{e^{(1-a)R}}{e^R - 1} 2\pi \longrightarrow 0$$

when $R \to \infty$

Putting all together and let $R \to \infty$:

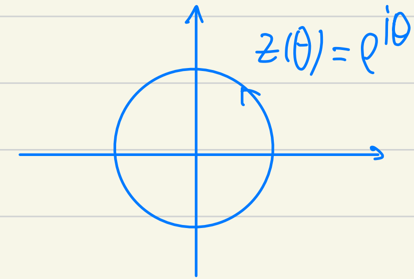
$$-2\pi i \cdot e^{a\pi i} = \int_C f(z)\, dz = \int_{C_0} f(z)\, dz + \int_{C_1} f(z)\, dz = \left(1 - e^{a\cdot 2\pi i}\right) \cdot \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx = -\frac{2\pi i \cdot e^{a\pi i}}{1 - e^{a\cdot 2\pi i}} = \frac{2\pi i}{e^{a\pi i} - e^{-a\pi i}} = \frac{2\pi i}{2i \cdot \sin(a\pi)} = \frac{\pi}{\sin(a\pi)}$$

$\square$

§ Compute Proper definite integral $\int_a^b f(x)\,dx$

More specifically, $\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$



Strategy: Let $z(\theta) = e^{i\theta}$, $\theta \in [0, 2\pi]$.

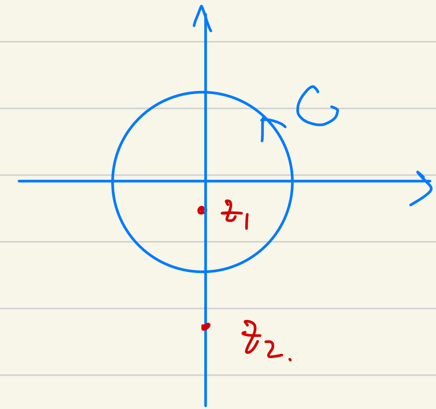
then $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{z - z^{-1}}{2i}$; $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{z + z^{-1}}{2}$

$dz = i \cdot e^{i\theta}\, d\theta = i \cdot z \cdot d\theta$

$$\Rightarrow \quad \int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta = \int_C F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{1}{iz}\, dz$$

where C is the unit circle.

Example 7: $\int_0^{2\pi} \frac{1}{1+a\sin\theta}\,d\theta \qquad -1<a<1,\ a\neq 0.$



$$= \int_C \frac{1}{1+a\cdot(z-z^{-1})/2i}\cdot\frac{1}{iz}\,dz$$

$$= \int_C \frac{2}{az^2+2iz-a}\,dz$$

Poles are the roots of $az^2+2iz-a=0$: $z_1=\frac{-1+\sqrt{1-a^2}}{a}i$, $z_2=\frac{-1-\sqrt{1-a^2}}{a}i$

As $-1<a<1$, $|z_2|>1$; as $z_1 z_2=-1$, $|z_1|<1$

Hence $\int_C f(z)\,dz = 2\pi i\,\mathrm{Res}(f, z_1) = 2\pi i\cdot\frac{2}{\underbrace{a\cdot(z_1-z_2)}_{2\sqrt{1-a^2}\cdot i}} = \frac{2\pi}{\sqrt{1-a^2}}$ $\square$