Note 5

§ Taylor and Laurent Series

Def: • For $z_n \in \mathbb{C}$, the (infinite) series $\sum_{n=0}^{\infty} z_n$ converges to the sum z if the sequence $S_N = \sum_{n=0}^{N} z_n$ of partial sum converges to z

$$\left(\lim_{N\to\infty} S_N = z \right)$$

• When a series does not converge, we say that it diverges.

• The series $\sum_{n=0}^{\infty} z_n$ converges absolutely if $\sum_{n=0}^{\infty} |z_n|$ converges.

allows "rearrangement"

Ex: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ Convergent but not absolute convergent.

Properties:

- Suppose $z_n = x_n + i y_n$ and $z = x + i y$

  Then $\sum_{n=0}^{\infty} z_n = z \iff \sum_{n=0}^{\infty} x_n = x$ and $\sum_{n=0}^{\infty} y_n = y$.

- Absolute convergence implies convergence.

  pf. $\sum_{n=0}^{\infty} |z_n| = \sum_{n=0}^{\infty} \sqrt{x_n^2 + y_n^2}$.

  Comparison Test.

  As $|x_n| \le \sqrt{x_n^2 + y_n^2}$ and $|y_n| \le \sqrt{x_n^2 + y_n^2}$, $\sum_{n=0}^{\infty} |x_n|$ and $\sum_{n=0}^{\infty} |y_n|$ converges

  $\Rightarrow \sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ converges

- Two standard tests on convergence of infinite series $\sum_{n=0}^{\infty} z_n$ Refined Version:

(1) Ratio test: If $L = \lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$ exists, then $\limsup_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$

- If $L<1$, then the series converges absolutely.
- If $L>1$, - - - - - - diverges
- If $L=1$, then the test gives no information.

(2). Root test: If $L = \lim_{n\to\infty} |z_n|^{\frac{1}{n}}$ exists, then $\limsup_{n\to\infty} |z_n|^{\frac{1}{n}}$

- If $L<1$, then the series converges absolutely
- If $L>1$, - - - - - diverges
- If $L=1$, - - - gives no information

Ex: $z_n = z^n$, $\sum_{n=0}^{\infty} z_n = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$

- By ratio test. $L = \lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = |z|$. $\Rightarrow$ $|z|<1$, abs. converges. $|z|>1$, diverges

- By root test. $L = \lim_{n\to\infty} |z_n|^{\frac{1}{n}} = |z|$. $\cdot \quad \cdot \quad \cdot \; ,$

Ex. $z_n = \frac{z^n}{n!}$. $\sum_{n=0}^{\infty} z_n = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots = e^z$.

- By ratio test, $L = \lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty} \frac{|z^{n+1}|/(n+1)!}{|z^n|/n!} = \lim_{n\to\infty} \frac{|z|}{n+1} = 0$

Hence, the series converges absolutely for all z.

lim sup and lim inf

Recall for a sequence $\{x_n\}$, $x_n \in \mathbb{R}$

$$\overline{\lim_{n\to\infty}}\, x_n = \limsup_{n\to\infty} x_n := \lim_{n\to\infty}\left(\sup_{m\geq n} x_m\right) = \inf_{n\geq 0}\left(\sup_{m\geq n} x_m\right)$$

$$\underline{\lim}\, x_n = \liminf_{n\to\infty} x_n := \lim_{n\to\infty}\left(\inf_{m\geq n} x_m\right) = \sup_{n\geq 0}\left(\inf_{m\geq n} x_m\right)$$

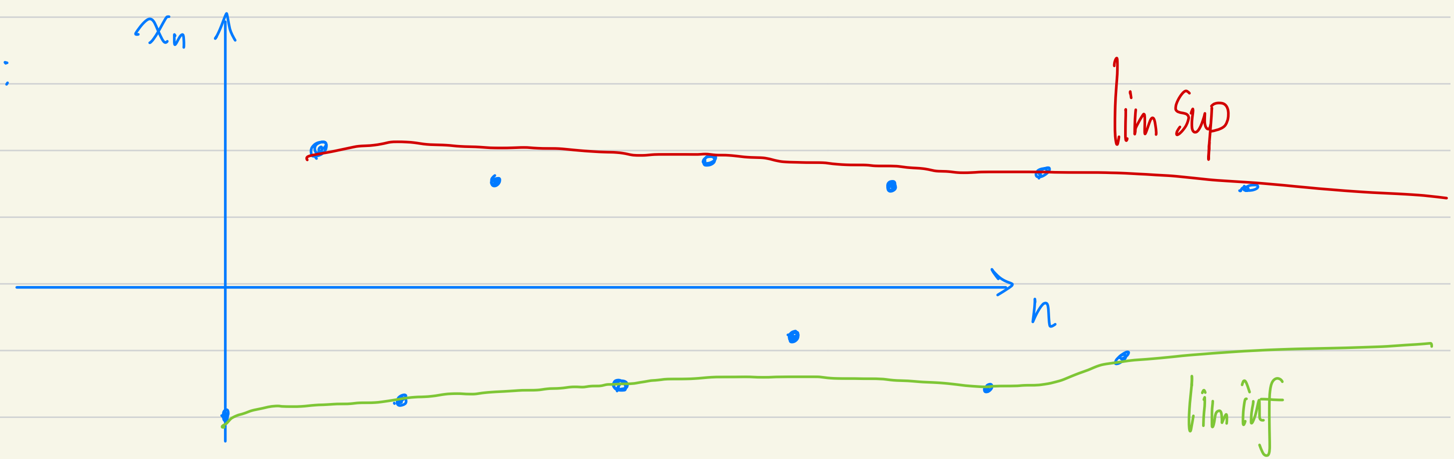
including ∞

- Unlike $\lim_{n\to\infty} x_n$ may not exist, $\limsup_{n\to\infty} x_n$, $\liminf_{n\to\infty} x_n$ always exist
- When $\lim_{n\to\infty} x_n$ exists, $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$

Ex. 1: $\{x_n\} = 0, 1, 0, 1, 0, 1 \cdots$

$\lim x_n$ does not exist, but $\limsup_{n\to\infty} x_n = 1$, $\liminf_{n\to\infty} x_n = 0$

Ex 2:



§ Power series:

are series of the form $\sum_{n=0}^{\infty} \underbrace{a_n z^n}_{= z_n}$ Variable (or more generally, $\sum_{n=0}^{\infty} a_n (z-z_0)^n$)

Theorem: Given a power series $f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$, there is a number $R \in [0, \infty]$, s.t.

the series converges absolutely if $|z-z_0| < R$ $= \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1}$

and - - - - - diverges " $|z-z_0| > R$

Def. The above R is called the radius of convergence (of the power series).

Disk $|z-z_0|<R$. disk of convergence.

Pf: Apply the Root Test (lim sup Version) to the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$.

$\Rightarrow$ · If $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \cdot |z-z_0| < 1$, then converges absolute

$$\Longleftrightarrow$$

$$|z-z_0| < \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1}$$

$\square$

Rmk: · When $R=\infty \Leftrightarrow \limsup |a_n|^{\frac{1}{n}} = 0$: Converges for all z.

When $R=0 \Leftrightarrow \limsup |a_n|^{\frac{1}{n}} = \infty$: diverges for all z.

Important Properties :

· Term by term differentiation : $f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$. radius of convergence $= R$

· - - - - integration : $\int_\gamma f(z)\,dz = \sum_{n=0}^{\infty} \int_\gamma a_n (z-z_0)^n \, dz$. $\gamma \subset$ disk of converg.

Ex. · $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \Rightarrow \left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n \cdot z^{n-1} = 1 + 2z + 3z^2 + \cdots$

· $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow (e^z)' = e^z = \sum_{n=1}^{\infty} \frac{n \cdot z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Consequently, $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ with $|z-z_0| < R$ is holomorphic !

Pf of "term by term differentiation": Consider $\lim\limits_{\Delta z \to \infty} \dfrac{f(z+\Delta z)-f(z)}{\Delta z}$

Suppose Δz is small enough s.t. both z and $z+\Delta z$ lie in the disk of conv.

Then $f(z) = \sum\limits_{n=0}^{\infty} a_n (z-z_0)^n$, $f(z+\Delta z) = \sum\limits_{n=0}^{\infty} a_n (z+\Delta z - z_0)^n$

Using "absolute convergence"

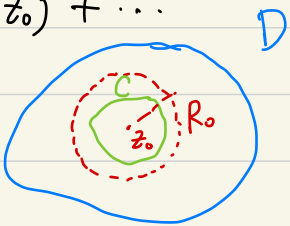
$$\Rightarrow \frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{\Delta z} \cdot \sum_{n=0}^{\infty} a_n \left((z+\Delta z - z_0)^n - (z-z_0)^n \right)$$

$$= \frac{1}{\cancel{\Delta z}} \cdot \sum_{n=0}^{\infty} a_n \cancel{\Delta z} \left((z+\Delta z - z_0)^{n-1} + (z+\Delta z - z_0)^{n-2} (z-z_0) + \cdots + (z-z_0)^{n-1} \right)$$

$$\Rightarrow f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \sum_{n=0}^{\infty} a_n \cdot n \cdot (z-z_0)^{n-1} \qquad \square$$

Conversely, holomorphic function $f(z)$ $\rightsquigarrow$ power Series ?

Def: Taylor Series about z_0: $f(z_0) + f'(z_0)\cdot(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots$



Maclaurin Series when $z_0 = 0$

Taylor's Theorem: Suppose $f(z)$ is holomorphic function in a region D, $z_0 \in D$.

Then $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, where $\boxed{a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$

Converges on any disk $|z-z_0| < R_0$ contained in D

Ex: (1) $e^z = 1 + z + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\quad |z| < \infty$

(2) $f(z) = z^3 \cdot e^{3z}$. As $e^{3z} = \sum_{n=0}^{\infty} \frac{(3z)^n}{n!}$. $f(z) = \sum_{n=0}^{\infty} \frac{3^n}{n!} \cdot z^{n+3}$

(3) $f(z) = \sin z$ $\quad |z| < \infty$

- Method 1: $f^{(n)}(0) = \sin^{(n)}(z)\Big|_{z=0} = \begin{cases} (-1)^m & n = 2m+1 \\ 0 & n \text{ even} \end{cases}$

- Method 2: $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$$= \frac{1}{2i}\left(\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} \right)$$

$i^n - (-i)^n = \begin{cases} 0 & n \text{ even} \\ 2i^n & n \text{ odd} \underset{2m+1}{\shortparallel} \end{cases}$

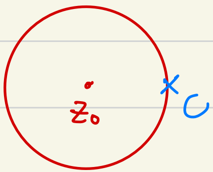
$$= \sum_{m=0}^{\infty} (-1)^m \cdot \frac{z^{2m+1}}{(2m+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

(4). $f(z) = \dfrac{1}{1-z} = 1+z+\dots+z^n + \dfrac{z^{n+1}}{1-z} \xrightarrow{n\to\infty} 0$ for $|z|<1$

$$\Rightarrow \quad \frac{1}{1-z} = 1+z+z^2+\dots = \sum_{n=0}^{\infty} z^n$$

In general, Consider $f(z) = \dfrac{1}{c-z}$ around $z_0 \neq c$

· Method 1: $f^{(n)}(z_0) = \dfrac{n!}{(c-z_0)^{n+1}}$



$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(c-z_0)^{n+1}}$$

Domain of f: $\mathbb{C} - \{c\}$

$\Rightarrow |z-z_0| < |c-z_0|$ disk of convergence.

· Method 2:

$$\frac{1}{C-z}$$

$$= \frac{1}{(C-z_0)-(z-z_0)}$$

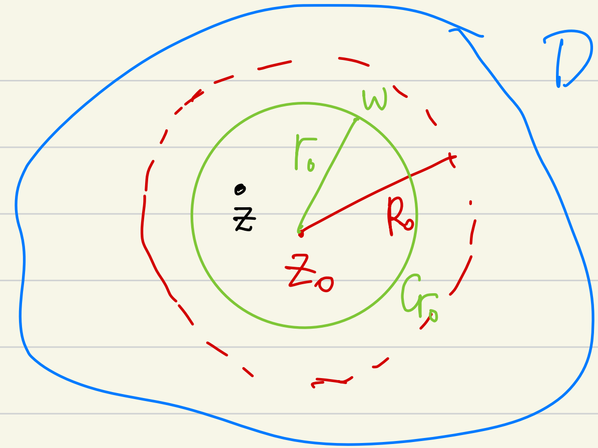
$$= \frac{1}{C-z_0} \cdot \frac{1}{1-\left(\frac{z-z_0}{C-z_0}\right)}$$

$$= \frac{1}{C-z_0} \cdot \left(1 + \frac{z-z_0}{C-z_0} + \frac{(z-z_0)^2}{(C-z_0)^2} + \cdots \right)$$

$$= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(C-z_0)^{n+1}}$$

Converges when $\left|\frac{z-z_0}{C-z_0}\right| < 1 \iff |z-z_0| < |C-z_0|$

- (Intuitive) Pf of Taylor's Theorem:



Let $C_{r_0} = \{ |w - z_0| = r_0 \}$, $r_0 < R_0$

$\forall z$ inside C_{r_0}, $f(z) = \frac{1}{2\pi i} \int_{C_{r_0}} \frac{f(w)}{w-z} dw$

Cauchy integral formula

Recall $\boxed{\frac{1}{w-z} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}}}$ for $|z-z_0| < |w-z_0| = r_0$

Thus $f(z) = \frac{1}{2\pi i} \int_{C_{r_0}} \sum_{n=0}^{\infty} f(w) \frac{(z-z_0)^n}{(w-z_0)^{n+1}} dw$

Cauchy Integral Formula

exchange $\int$ and Σ need justification

$$= \sum_{n=0}^{\infty} (z-z_0)^n \cdot \underbrace{\frac{1}{2\pi i} \int_{C_{r_0}} \frac{f(w)}{(w-z_0)^{n+1}} dw}_{} = \frac{f^{(n)}(z_0)}{n!} = a_n$$

• (Rigorous) Proof :

Rewrite : $$\frac{1}{w-z} = \frac{1}{w-z_0} \cdot \frac{1}{1-\frac{z-z_0}{w-z_0}}$$

$$\frac{1}{1-A} = 1+A+\cdots+A^n+\frac{A^{n+1}}{1-A}$$

$$= \frac{1}{w-z_0}\left(1+\frac{z-z_0}{w-z_0}+\cdots+\left(\frac{z-z_0}{w-z_0}\right)^N+\frac{\left(\frac{z-z_0}{w-z_0}\right)^{N+1}}{1-\frac{z-z_0}{w-z_0}}\right)$$

$$= \sum_{n=0}^{N}\frac{(z-z_0)^n}{(w-z_0)^{n+1}} + \frac{(z-z_0)^{N+1}}{(w-z)\cdot(w-z_0)^{N+1}}$$

Finite sum, do integration term by term:

• $$\frac{1}{2\pi i}\int_{C_{r_0}}\frac{f(w)}{(w-z_0)^{n+1}}\cdot(z-z_0)^n\,dw = \frac{f^{(n)}(z_0)}{n!}\cdot(z-z_0)^n$$

- The last integral is $\frac{1}{2\pi i} \int_{C_{r_0}} f(w) \cdot \frac{(z-z_0)^{N+1}}{(w-z)\cdot(w-z_0)^{N+1}} \, dw$ $\longrightarrow 0$ when $N \to \infty$

(Justification: Let $\max_{w \in C_{r_0}} \left| \frac{f(w)}{w-z} \right| = M$

$\max_{w \in C_{r_0}} \left| \frac{z-z_0}{w-z_0} \right| = C < 1$,

then $| \cdots | \le M \cdot C^{N+1} \text{length}(C_{r_0})$ $\longrightarrow 0$ when $N \to \infty$)

Limit $= 0$ when $N \to \infty$.

Thus, $\lim_{N \to \infty} \sum_{n=0}^{N} a_n (z-z_0)^n = f(z)$.

$\square$

§ Zeroes

Suppose $f(z)$ holomorphic on $|z-z_0|<R_0$ and f is not identically 0.

Taylor's thm: $$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

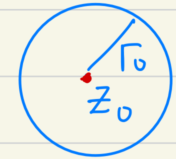
Def: z_0 is a zero of f if $f(z_0)=0$.

the order of the zero of f at z_0 is k.

If $a_0=a_1=\cdots=a_{k-1}=0$, $a_k\neq 0$.

Then $$f(z)=(z-z_0)^k\cdot(a_k+a_{k+1}(z-z_0)+\cdots)$$

Theorem: If $f \not\equiv 0$ holomorphic, then the zeroes of f are isolated.



i.e., $\forall$ z_0 zero, there's no zero in $0 < |z - z_0| < r_0$

pf: Let $g(z) = a_k + a_{k+1}(z - z_0) + \cdots$

$g(z_0) = a_k \neq 0$. $\Longrightarrow$ $g(z) \neq 0$ in a small neighborhood

$\square$

Corollary: If two holomorphic functions on D $f(z) = g(z)$ over an open set

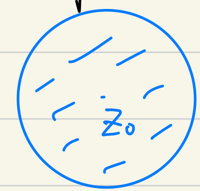
or, in general, a set with accumulation pt.

then $f = g$ throughout D

(Local determines Global !)

§ Laurent Series

- Power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n \rightsquigarrow f(z)$ holomorphic in disk of convergence



$|z-z_0| < R$

- Laurent Series:

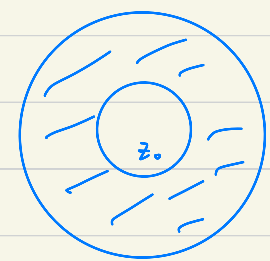
$$\underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{} \rightsquigarrow f(z) \text{ holomorphic in Annulus}$$

$\frac{1}{r} < |z-z_0| < R$

Converge: $|z-z_0| < R$

$\frac{1}{|z-z_0|} < r$

$\iff |z-z_0| > \frac{1}{r}$



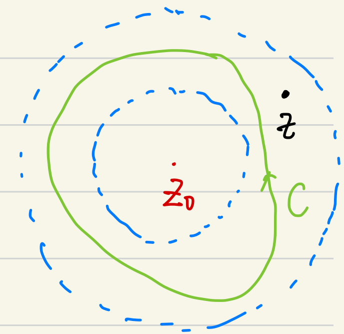
Thm: (Laurent Series) Suppose $f(z)$ is holomorphic on the annulus A: $R_1 < |z-z_0| < R_2$.

Then $\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}$ for $z \in A$

where the coefficients $a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{n+1}}\, dw$

$\forall\, C \subset A$

and $b_n = \frac{1}{2\pi i} \int_C f(w) \cdot (w-z_0)^{n-1}\, dw$



In particular,

- $\boxed{\sum_{n=0}^{\infty} a_n (z-z_0)^n}$ Def: analytic / regular part of Laurent series. Converges to an analytic function for $|z-z_0| < R_2$
- $\boxed{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}$ Def: Singular / principal part of Laurent series. . . . . $|z-z_0| > R_1$

Examples:

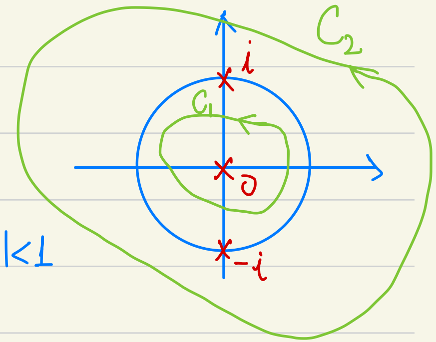
(1). $f(z)$ holomorphic throughout the disk $|z-z_0|<R_2$

then $$a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw = \frac{1}{n!} f^{(n)}(z_0).$$

$$b_n = \frac{1}{2\pi i}\int_C f(w)\cdot(w-z_0)^{n-1}\,dw = 0$$

So Laurent Series = Taylor Series.

(2). $f(z) = \frac{1}{z \cdot (z^2+1)}$ Singularities at $0, \pm i$



A_1: $0 < |z| < 1$: $\frac{1}{1+z^2} = 1 - z^2 + z^4 - \dots = \sum_{n=0}^{\infty} (-1)^n \cdot z^{2n}, \quad |z| < 1$

So $f(z) = \sum_{n=0}^{\infty} (-1)^n \cdot z^{2n-1} = \underbrace{\frac{1}{z}}_{\text{principal}} + \underbrace{\sum_{n=1}^{\infty} (-1)^n z^{2n-1}}_{\text{analytic} = \frac{-z}{z^2+1}} \quad 0 < |z| < 1$

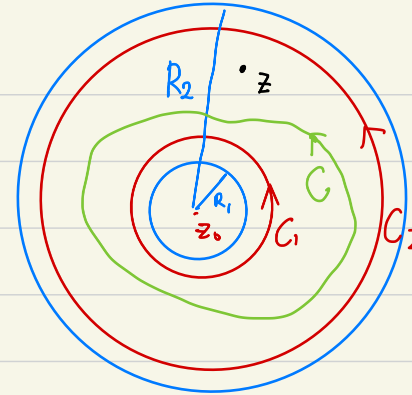
A_2: $1 < |z| < \infty$: $\frac{1}{1+z^2} = \frac{1}{z^2} \cdot \left(\frac{1}{\frac{1}{z^2} + 1} \right) = \frac{1}{z^2} \cdot \left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \dots \right), \quad |z| > 1$

So $f(z) = \frac{1}{z^3} \cdot \left(1 - \frac{1}{z^2} + \dots \right) = \underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{z^{2n+1}}}_{\text{principal}}$

Note: Laurent series depends on z_0 AND <u>the annulus</u>!

Proof of Thm: Suppose $C_1: |z-z_0|=r_1$

$C_2: |z-z_0|=r_2$

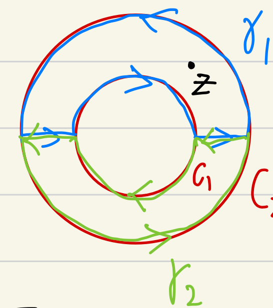


s.t. $R_1 < r_1 < r_2 < R_2$, C, z in the annulus $r_1 < |z-z_0| < r_2$.

Claim. $$\boxed{f(z) = \frac{1}{2\pi i}\int_{C_2 - C_1} \frac{f(w)}{w-z}\,dw} = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw$$

pf: $f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}\,dw$, $0 = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}\,dw$

Cauchy integral formula Cauchy theorem



$\gamma_1 \cup \gamma_2 = C_2 \cup -C_1$.

$\square$

- For integral over C_2. $\left|\frac{z-z_0}{w-z_0}\right| < 1$

Write $\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)}$

$$= \frac{1}{w-z_0} \cdot \frac{1}{1-\frac{z-z_0}{w-z_0}}.$$

$$= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}}$$

Hence, $$\frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw = \sum_{n=0}^{\infty} (z-z_0)^n \cdot \underbrace{\frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-z_0)^{n+1}}\,dw}_{\parallel}$$

justification needed

$$a_n := \int_{C} \frac{f(w)}{(w-z_0)^{n+1}}\,dw$$

Cauchy thm

- For integral over C_1. $\left|\frac{w-z_0}{z-z_0}\right| < 1$

Write $\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)}$

$$= \frac{1}{z-z_0} \cdot \frac{1}{\frac{w-z_0}{z-z_0} - 1}$$

$$= -\sum_{n=0}^{\infty} \frac{(w-z_0)^n}{(z-z_0)^{n+1}} = \sum_{n=1}^{\infty} \frac{(w-z_0)^{n-1}}{(z-z_0)^n}$$

Hence, $\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw = -\sum_{n=1}^{\infty} (z-z_0)^{-n} \cdot \underbrace{\frac{1}{2\pi i}\int_{C_1} f(w)\cdot(w-z_0)^{n-1}\,dw}_{\substack{\| \\ b_n := \int_C f(w)(w-z_0)^{n-1}\,dw}}$ Cauchy thm

justification needed

$\square$