THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 Honours Algebraic Structures 2023-24
Tutorial 7 Solutions
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1. (a) We have $\sigma = (1537624)$ and $\tau = (1426735)$.

   (b) Both σ and τ are 7-cycles so they have order 7.

   (c) One can easily check that $\sigma \circ \tau(k) = k$ holds for all $k \in \{1, 2, ..., 7\}$, therefore $\sigma \circ \tau = \mathrm{id}$.

   (d) From (c), we have $\tau = \sigma^{-1}$, so $\langle \sigma, \tau \rangle = \langle \sigma \rangle = \{\mathrm{id}, \sigma, \sigma^2, ..., \sigma^6\}$. This is not a normal subgroup, for example $(12)\sigma(12) = (2537614) \neq \sigma$.

2. Note that $\psi : G \to H/K$ is the composition of $\varphi : G \to H$ with $\pi : H \to H/K$ where π is the canonical projection defined by $\pi(h) = hK \in H/K$. Indeed, $\pi \circ \varphi(g) = \pi(\varphi(g)) = \varphi(g)K = \psi(g)$. Therefore ψ is a group homomorphism. The surjectivity of ψ then follows from the surjectivity of both φ and π. Alternatively, one can simply note that for any $hK \in H/K$, we may find some $g \in G$ so that $\varphi(g) = h$, so that $\psi(g) = \varphi(g)K = hK$.

   Now it suffices to show that $\ker \psi = \varphi^{-1}(K)$. Indeed,

$$\begin{aligned} g \in \ker \psi &\iff \psi(g) = K \in H/K \\ &\iff \varphi(g)K = K \\ &\iff \varphi(g) \in K \\ &\iff g \in \varphi^{-1}(K). \end{aligned}$$

   Hence, by the first isomorphism theorem, we get $G/\varphi^{-1}(K) \cong H/K$.

3. (a) Let $d = |(x, y)|$, note that $(x, y)^d = (e_H, e_K)$, so that $x^d = e_H$ and $y^d = e_K$, in particular, $m = |x|$ divides d and $n = |y|$ divides d, so that $\mathrm{lcm}(m, n)$ also divides d. On the other hand, $\mathrm{lcm}(m, n)$ is by definition a multiple of m and n, so $(x, y)^{\mathrm{lcm}(m,n)} = (x^{\mathrm{lcm}(m,n)}, y^{\mathrm{lcm}(m,n)}) = (e_H, e_K)$. Thus the order d must divide $\mathrm{lcm}(m, n)$. This proves their equality.

   (b) Note that $|D_p| = 2p$ is a product of two prime numbers, so if $D_p \cong H \times K$ for some non-trivial groups, then $|H| = p$ and $|K| = 2$; or $|H| = 2$ and $|K| = 2$. Without loss of generality, assume the former, then since H, K have prime orders, they are cyclic groups. Let $x \in H$ and $y \in K$ be generators of their respective groups, then $|x| = p$ and $|y| = 2$, and by (a), we know that (x, y) has order $\mathrm{lcm}(p, 2) = 2p$. Since D_p has order $2p$, the image of (x, y) under the isomorphism $D_p \cong H \times K$ also has order $2p$, this would imply that D_p is cyclic, which is a contradiction.

4. Note that 600 factorizes as $2^3 \cdot 3^1 \cdot 5^2$. Therefore, by classification theorem, we have to consider partitions of the tuples of integers $(3, 1, 2)$, as follows:

(i) Partition $(3,1,2)$: $\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_{25} \cong \mathbb{Z}_{600}$.

(ii) Partition $(1+2,1,2)$: $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_{25} \cong \mathbb{Z}_2 \times \mathbb{Z}_{300}$.

(iii) Partition $(1+1+1,1,2)$: $(\mathbb{Z}_2)^3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{150}$.

(iv) Partition $(3,1,1+1)$: $\mathbb{Z}_8 \times \mathbb{Z}_3 \times (\mathbb{Z}_5)^2 \cong \mathbb{Z}_5 \times \mathbb{Z}_{120}$.

(v) Partition $(1+2,1,1+1)$: $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times (\mathbb{Z}_5)^2 \cong \mathbb{Z}_{10} \times \mathbb{Z}_{60}$.

(vi) Partition $(1+1+1,1,1+1)$: $(\mathbb{Z}_2)^3 \times \mathbb{Z}_3 \times (\mathbb{Z}_5)^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_{10} \times \mathbb{Z}_{30}$.

Very importantly, remember that $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ if and only if $\gcd(m,n) = 1$. So there are many ways to write down the same group, for example (i) can be equivalently expressed as $\mathbb{Z}_{600} \cong \mathbb{Z}_3 \times \mathbb{Z}_{200} \cong \mathbb{Z}_8 \times \mathbb{Z}_{75} \cong \mathbb{Z}_{24} \times \mathbb{Z}_{25} \cong \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$.

5. Consider the sum $\sum_{g\in G} \varphi(g) \in \mathbb{C}$, since φ is a nontrivial homomorphism, there exists some $h \in G$ so that $\varphi(h) \neq 1$, then we have

$$\varphi(h)\sum_{g\in G}\varphi(g) = \sum_{g\in G}\varphi(h)\varphi(g) = \sum_{g\in G}\varphi(hg) = \sum_{hg=g'\in G}\varphi(g').$$

If we write $x = \sum_{g\in G}\varphi(g)$, the above equality reads $\varphi(h)x = x$. Since $\varphi(h) \neq 1$, we must have $x = 0$.

6. For any $h \in H, k \in K$ note that $khk^{-1} \in H$ and $hk^{-1}h^{-1} \in K$ by normality, therefore $khk^{-1}h^{-1}$ is in $H \cap K = \{e\}$, so that $hk = kh$. Now we define $\varphi : H \times K \to G$ by $\varphi(h,k) = hk$. This map is a group homomorphism because

$$\varphi((h_1,k_1)\cdot(h_2,k_2)) = \varphi(h_1h_2,k_1k_2) = h_1h_2k_1k_2 = h_1k_1h_2k_2 = \varphi(h_1,k_1)\varphi(h_2,k_2)$$

and

$$\varphi((h,k)^{-1}) = \varphi(h^{-1},k^{-1}) = h^{-1}k^{-1} = k^{-1}h^{-1} = \varphi(h,k)^{-1}.$$

Note that surjectivity of φ is true by assumption, so it remains to check injectivity: let $\varphi(h,k) = hk = e \in G$, then we have $h = k^{-1}$, therefore $h \in H \cap K = \{e\}$, so $h = k = e$.