THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 Honours Algebraic Structures 2023-24
Tutorial 7 Problems (Revision Exercise)
11th March 2024

- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. Consider the following elements in S_7.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 4 & 7 & 1 & 3 & 2 & 6 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 5 & 2 & 1 & 7 & 3 \end{pmatrix}.$$

   (a) Express σ and τ as a product of disjoint (commuting) cycles.

   (b) Determine the order of σ and τ in S_7.

   (c) Find $\sigma \circ \tau$.

   (d) Determine the subgroup $\langle \sigma, \tau \rangle \leq S_7$ explicitly by listing all the elements in this subgroup. Is this subgroup normal in S_7?

2. Let $\varphi : G \to H$ be a surjective group homomorphism, and $K \trianglelefteq H$, prove that $G/\varphi^{-1}(K) \cong H/K$ by considering the map $\psi : G \to H/K$ defined by $\psi(g) = \varphi(g)K \in H/K$.

3. (a) Let H, K be groups, suppose that $x \in H, y \in K$ are of orders m and n respectively, show that the element $(x, y) \in H \times K$ has order $\text{lcm}(m, n)$.

   (b) Let p be an odd prime number, consider D_p the p-th dihedral group, using part (a) or otherwise, prove that D_p is not isomorphic to a product of two nontrivial groups.

4. List all abelian groups of order 600.

5. Let G be a finite group, and $\varphi : G \to \mathbb{C}^\times$ be a nontrivial group homomorphism, prove that $\sum_{g \in G} \varphi(g) = 0$.

6. Let G be a group, $H, K \trianglelefteq G$ be normal subgroups such that $H \cap K = \{e\}$. Suppose that any $g \in G$ can be expressed as a product $g = hk$ where $h \in H, k \in K$, prove that $G \cong H \times K$. (Hint: first show that $hk = kh$ for any elements $h \in H, k \in K$.)