THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 Honours Algebraic Structures 2023-24
Tutorial 6 Problems
26th February 2024

- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. Prove that $\mathbb{R}/2\pi\mathbb{Z} \cong U$ where U is the multiplicative group of unit length elements in $\mathbb{C}$.

2. Prove the remark in P.28 from the lecture notes: if $\varphi : G \to G'$ is a surjective homomorphism, then G is cyclic $\implies G'$ is cyclic and G is abelian $\implies G'$ is abelian.

3. (a) Let G be a group, prove that for any element $g \in G$, there is a group homomorphism $\varphi : \mathbb{Z} \to G$ such that $\varphi(1) = g$.

   (b) Suppose that $\varphi : D_n \to G$ is a group homomorphism, what relations do the elements $\varphi(r)$ and $\varphi(s)$ have to satisfy?

4. Prove that $\mathbb{Q} \ncong \mathbb{Z}$ and $\mathbb{Q} \ncong \mathbb{R}$.

5. Let G be a group, consider $\varphi : G \to \mathrm{Sym(G)}$, where $\mathrm{Sym}(G)$ is the group of bijection from G to itself regarded as a set, and φ is defined by $\varphi(g) : G \to G$ the left multiplication: $\varphi(g)(x) = gx$. Prove that G is an injective homomorphism.

6. (a) Let $H \leq G$ be a proper subgroup of finite index, say $[G : H] = n$, let X be the set of left H cosets in G, denote $\mathrm{Sym}(X)$ the symmetric group on X, i.e. the group of bijective functions from X to itself. Define the homomorphism $\varphi : G \to \mathrm{Sym}(X)$ by the bijection $\varphi(g) : X \to X$ by sending a coset aH to $(ga)H$. Show that $\ker(\varphi) \leq H$.

   (b) Using part (a) and the first isomorphism theorem, prove that there exists a normal subgroup $\tilde{H} \trianglelefteq G$ such that $n \leq [G : \tilde{H}] \leq n!$.

   (c) Explain why an infinite group G with a proper finite index subgroup must have a nontrivial proper normal subgroup.

7. Recall that $Z(G) = \{x \in G : \ gx = xg \text{ for all } g \in G\}$ be the center of G, prove that $G/Z(G) \cong \mathrm{Inn}(G)$ using the first isomorphism theorem. (See HW5 compulsory Q7 for the definition of $\mathrm{Inn}(G)$.)

8. What is $\mathrm{Aut}(\mathbb{Z})$?

9. Let m, n be positive integers that are coprime, prove that there does not exist any homomorphism $\mathbb{Z}_m \to \mathbb{Z}_n$.