THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 Honours Algebraic Structures 2023-24
Tutorial 5 Solutions
19th February 2024

- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. $(a) \implies (b)$ holds by definition.

   $(b) \implies (c)$ is true because $aH \subset Ha \iff$ for all $h \in H$ there exists $h' \in H$ such that $ah = h'a \iff$ for all $h \in H$ there exists $h' \in H$ so that $aha^{-1} = h' \iff aHa^{-1} \subset H$.

   $(c) \implies (d)$: Let $x \in H$ be any element, then $axa^{-1} \in H$ for arbitrary $a \in G$, so that $C_x = \{axa^{-1} : a \in G\} \subset H$. This implies that $\bigcup_{x\in H} C_x \subset H$, on the other hand $H \subset \bigcup_{x\in H} C_x$ since any $x \in H$ would have $x \in C_x$.

   $(d) \implies (a)$: Let H be a union of conjugacy classes, say $H = \bigcup_{x\in I} C_x$. Since conjugacy classes are equivalence classes coming from the equivalence relation $x \sim axa^{-1}$. This implies that H is in fact a disjoint union. Then consider an element $ah \in aH = \bigsqcup_{x\in I} aC_x$, then $h \in C_x$ for some $x \in I$. By definition, we have $aha^{-1} \in C_x$, therefore $aha^{-1}a = ah \in C_x a \subset \bigcup_{x\in I} C_x a = Ha$. So we have shown that $aH \subset Ha$. The other direction is similar.

   $(e) \implies (a)$ is true by corollary from lecture notes.

   $(a) \implies (e)$: If $H \trianglelefteq G$, then the canonical projection $\pi : G \to G/H$ defined by $a \mapsto aH$ is a well-defined group homomorphism, with $\ker \pi = H$. (See 6.2 of lecture notes.)

2. (a) For any $a \in G$, $a\{e\} = \{a\} = \{e\}a$, so $\{e\}$ is normal. As for $a \in G$, we have $aG = G = Ga$ for any $a \in G$ since multiplication on the left and on the right define bijection functions from G to itself.

   (b) The simplest way is to consider $aZ(G)a^{-1} = \{axa^{-1} : x \in Z(G)\} = \{x : x \in Z(G)\} = Z(G)$ for any $a \in G$. Therefore criterion (c) of Q1, $Z(G)$ is normal.

   It is also possible to prove it using the criterion (e) of Q1. We will need to construct a homomorphism $\varphi : G \to G'$ such that $\ker \varphi = Z(G)$. The idea is to use a natural conjugate "action" of G. For each $g \in G$, we may define a function $\varphi_g : G \to G$ by $\varphi_g(x) = gxg^{-1}$. This is a bijective function since its inverse can be easily seen to be $\varphi_g^{-1} = \varphi_{g^{-1}}$, as $\varphi_{g^{-1}} \circ \varphi_g(x) = g^{-1}gxg^{-1}g = x$, and $\varphi_g \circ \varphi_{g^{-1}}(x) = gg^{-1}xgg^{-1} = x$. The functions φ_g also satisfies that $\varphi_{gh}(x) = ghx(gh)^{-1} = ghxh^{-1}g^{-1} = \varphi_g \circ \varphi_h(x)$. These properties implies that the map $\varphi : G \to \mathrm{Sym}(G)$ defined by $g \mapsto \varphi_g$ is a group homomorphism, where $\mathrm{Sym}(G)$ is the symmetric group on the underlying set of G (i.e. the group of bijection function from G to G, equipped with function composition.)

   We claim that $Z(G) = \ker \varphi$. Indeed, $g \in Z(G) \iff gxg^{-1} = x$ for any $x \in G \iff \varphi_g(x) = \mathrm{id}(x)$ for any $x \in G \iff \varphi_g = \mathrm{id} \iff g \in \ker \varphi$. Thus $Z(G)$ is normal.

(c) Recall that $[G : H]$ is defined as the number (cardinality) of left cosets of H in G. First, we shall prove that the number of left cosets is the same as the number or right cosets. Let L and R be the set of left and right cosets of H in G respectively, we define a function $f : L \to R$ by $f(aH) = Ha^{-1}$. We claim that this is a well-defined bijective function, thus showing that $|L| = |R|$. If a and a' represent the same left cosets, i.e. $aH = a'H$, then there exists some $h \in H$ so that $a' = ah$, then $f(a'H) = Ha'^{-1} = Hh^{-1}a^{-1} = Ha^{-1} = f(aH)$, so that f is well-defined. It is injective because

$$\begin{aligned} f(aH) = f(a'H) &\iff Ha^{-1} = Ha'^{-1} \\ &\iff \text{there exists some } h \in H \text{ so that } a'^{-1} = ha^{-1} \\ &\iff \text{there exists some } \tilde{h} \text{ so that } a' = a\tilde{h} \\ &\iff aH = a'H. \end{aligned}$$

And it is surjective since for any right coset $Hb \in R$, $f(b^{-1}H) = Hb$. Thus f is bijective and $|L| = |R|$.

In our case, that means that there are precisely two left cosets and two right cosets of H in G. Since cosets define a partition of G, that means that if $a \notin H$, then $aH \neq H$ and $Ha \neq H$, so that aH and Ha must both be the complement of H, $G \setminus H$, in particular they are equal. And if $a \in H$, then $aH = H = Ha$. Therefore we have proven that H is normal.

(d) If $\{H_i\}_{i \in I}$ are normal subgroups, then for any $a \in G$, we have $aH_i = H_i a$. Now for $\bigcap_{i \in I} H_i$, note that for any $a \in G$,

$$a \bigcap_{i \in I} H_i := \{ah : h \in H_i \text{ for all } i \in I\} = \bigcap_{i \in I} aH_i = \bigcap_{i \in I} H_i a = \left(\bigcap_{i \in I} H_i \right) a.$$

So $\bigcap_{i \in I} H_i$ is normal.

We can also prove the normality by constructing a homomorphism. Recall that since H_i are normal, we have canonical projections $\pi_i : G \to G/H_i$ defined by $\pi_i(a) = aH_i$ such that $\ker \pi_i = H_i$. Consider $\pi : G \to \prod_{i \in I} G/H_i$ defined by $\pi(a) = (\pi_i(a))_{i \in I} = (aH_i)_{i \in I}$. We claim that $\ker \pi = \bigcap_{i \in I} H_i$. Indeed, $a \in \ker \pi$ precisely when $\pi(a) = e \in \prod_{i \in I} G/H_i$, but the identity e in the product is just the product of identities, so that $\pi_i(a) = e_i \in G/H_i$ for all $i \in I$. This happens exactly when $a \in \ker \pi_i$ for all $i \in I$, i.e. $a \in \bigcap_{i \in I} \ker \pi_i = \bigcap_{i \in I} H_i$.

(e) If $K \leq H \leq G$ and $K \trianglelefteq G$, then for all $a \in G$, $aK = Ka$. In particular, for any $a \in H \leq G$, $aK = Ka$, so K is also normal in H.

We can also prove it using kernel. Since $K \trianglelefteq G$, we have $\pi : G \to G/K$ with $\ker \pi = K$. Clearly, the restriction $\pi|_H : H \to G/K$ is still a homomorphism, with $\ker(\pi|_H) = K$, thus $K \trianglelefteq H$.

(f) For any $a \in G_1$, $b \in G_2$, we have $(a, b)H \times K = aH \times bK = Ha \times Kb = H \times K(a, b)$, so that $H \times K$ is normal.

Alternatively, consider canonical projections $\pi_1 : G_1 \to G_1/H$ and $\pi_2 : G_2 \to G_2/K$, we may define $\pi : G_1 \times G_2 \to G_1/H \times G_2/K$ by $\pi(a, b) = (\pi_1(a), \pi_2(b))$.

Then we have

$$\begin{aligned}\ker \pi &= \{(a, b) :\ (\pi_1(a), \pi_2(b)) = (aH, bK) = (H, K) \in G_1/H \times G_2/K\} \\ &= \{(a, b) :\ a \in H, b \in K\} \\ &= H \times K.\end{aligned}$$

3. Let G/H be the set of left cosets of H in G and similar for G/K and H/K (we did not assume normality of H in G etc, here G/H is only a set, not necessarily with a group structure.) Define a function $f : G/K \to G/H$ by $f(aK) = aH$. This is well-defined since if $aK = a'K$ then $a' = ak$ for some $k \in K$, so that $a'H = akH = aH$ as $k \in H$. We claim that f is surjective, with $|f^{-1}(aH)| = [H : K]$ for any $aH \in G/H$. Thus, we have

$$\begin{aligned}[][G : K] = |G/K| &= \left| \bigsqcup_{aH \in G/H} f^{-1}(aH) \right| \\ &= \sum_{aH \in G/H} |f^{-1}(aH)| \\ &= \sum_{aH \in G/H} [H : K] \\ &= [G : H][H : K].\end{aligned}$$

Now we prove the claim. Clearly f is surjective, since for any $aH \in G/H$, we may take $aK \in G/K$, then $f(aK) = aH$ by definition. Now fix some $aH \in G/K$ and consider $f^{-1}(aH)$. We have $bK \in f^{-1}(aH)$ when $f(bK) = bH = aH$. This happens precisely when $b = ah$ for some $h \in H$. Thus, we have

$$f^{-1}(aH) = \{ahK \in G/K :\ h \in H\}.$$

Now consider the function $s : f^{-1}(aH) \to H/K$ where we send $ahK \in f^{-1}(aH)$ to $s(ahK) = hK$. This is well-defined because if $ahK = ah'K$, then $ah' = ahk$ for some $k \in K$, so that $h' = hk$, which implies that $h'K = hK$. Note that s is also an injective function, since $hK = s(ahK) = s(ah'K) = h'K$ precisely when $h' = hk$, which implies $ah = ah'k$, so that $ahK = ah'K$. Surjectivity of s follows from the definition directly. Thus s is a bijection and establishes $|f^{-1}(aH)| = |H/K| = [H : K]$.

The converse is false. Consider $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $\langle (1, 1) \rangle$ is a normal subgroup that is not a product.

4. If G is a cyclic group, take $a \in G$ be a generator. Then any $g \in G$ can be written as $g = a^k$ for some $k \in \mathbb{Z}$. Now let $gH \in G/H$ be any element, then $gH = a^k H = (aH)^k$ for some $k \in \mathbb{Z}$. Thus aH is a generator of G/H and G/H is cyclic.

5. If both H and G/H are cyclic, let $a \in H$ and $bH \in G/H$ be generators. We claim that $\langle a, b \rangle = G$. Let $g \in G$ be any element, then g is in the left coset gH, so there exists some k so that $gH = (bH)^k = b^k H$. Therefore, there is some $h \in H$ such that $g = b^k h$. Now H is generated by a, so there is some l so that $h = a^l$. Putting these together yields $g = b^k h = b^k a^l$.