THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 Honours Algebraic Structures 2023-24
Tutorial 5 Problems
19th February 2024

- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. Let $H \leq G$, prove that the following statements are equivalent.
   (a) $H \trianglelefteq G$.
   (b) $aH \subseteq Ha$ for all $a \in G$.
   (c) $aHa^{-1} \subseteq H$ for all $a \in G$.
   (d) H is a union of conjugacy classes. (See Q4 of tutorial 4.)
   (e) $H = \ker \varphi$ for some group homomorphism $\varphi : G \to G'$.

2. Prove the following properties about normal subgroups.
   (a) $\{e\} \trianglelefteq G$ and $G \trianglelefteq G$.
   (b) The center $Z(G) := \{x \in G : \ gx = gx \ \forall g \in G\}$ is a normal subgroup of G. (See Q4 of tutorial 3.)
   (c) If $H \leq G$ is an index 2 subgroup, i.e. $[G : H] = 2$, then H is normal in G.
   (d) If $\{H_i\}_{i \in I}$ is an arbitrary collection of normal subgroups of G, then $\bigcap_{i \in I} H_i$ is normal in G.
   (e) If $K \leq H \leq G$ and $K \trianglelefteq G$, then $K \trianglelefteq H$.
   (f) If $H \trianglelefteq G_1$ and $K \trianglelefteq G_2$, then $H \times K \trianglelefteq G_1 \times G_2$. (Is the converse true? I.e. is every normal subgroups of $G_1 \times G_2$ equal to a product of normal subgroups?)

3. Let $K \leq H \leq G$ such that $[G : H]$ and $[H : K]$ are finite, show that $[G : K] = [G : H][H : K]$. (Hint: try to construct a function from the set of left cosets of K in G to the set of left cosets of H in G, with preimages overe each element having cardinality equals to $[H : K]$.)

4. Let G be a cyclic group, prove that G/H is again cyclic for any normal subgroup $H \trianglelefteq G$.

5. Let H be a normal subgroup of G such that both H and G/H are cyclic, prove that G is generated by two elements.