THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 Honours Algebraic Structures 2023-24
Tutorial 3 Problems
29th January 2024

- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. Let $H \leq G$ be a subgroup, prove that for any $g \in G$, $gHg^{-1} := \{gxg^{-1} : x \in H\}$ is also a subgroup of G. Moreover, if H is finite, then H and gHg^{-1} have the same order.

2. Let H, K be subgroups of G, are $H \cap K$ and $H \cup K$ subgroups of G? Prove it if it is true, otherwise provide a counter-example.

3. Recall from HW1 compulsory Q4, we can construct the product group $G_1 \times G_2$ of two groups G_1, G_2. Prove that if $H \leq G_1$ and $K \leq G_2$ are subgroups, then $H \times K \subset G_1 \times G_2$ is a subgroup.

4. Let G be a group, and define the subset $Z := \{g \in G : gx = xg, \forall x \in G\}$, prove that Z is always a subgroup of G, this subgroup is called the center of G.

5. Let $H \subset G$ be a subgroup, define the subset $N_G(H) := \{g \in G : gHg^{-1} = H\}$, prove that $N_G(H)$ is a subgroup. This subgroup is called the normalizer of H in G.

6. Recall that a subset H of G is a subgroup if and only if the following two conditions hold:
   (a) $x, y \in H$ implies $xy \in H$, and
   (b) $x \in H$ implies $x^{-1} \in H$.

   Give a counter-example to show that (b) cannot be dropped, i.e. find a subset H of some group G that is closed under multiplication, but H is not a subgroup of G.

7. Following Q5, prove that (b) follows from (a) in the case where G is finite.

8. Recall that every subgroup of a cyclic subgroup is cyclic. Determine whether the following statement is true: if every *proper* subgroup of G is cyclic, then G is cyclic.

9. List all subgroups of $\mathbb{Z}_8, \mathbb{Z}_{11}$ and $\mathbb{Z}_{12}$.

10. Prove that a group G is finite if and only if it has finitely many subgroups.

11. Show that it is impossible for a group to be a union of two proper subgroups.

12. Let G be a finite group of order n such that every non-identity element has order 2.

    (a) Show that G is abelian.

    (b) Let H be a proper subgroup of G (a subgroup that is not equal to the whole of G), take $g \notin H$, show that $H \cup gH$ is a subgroup of G.

    (c) Show that $|H \cup gH| = 2|H|$.

    (d) Prove that $n = 2^k$ for some integer k.