THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 Honours Algebraic Structures 2023-24
Tutorial 2 Problems
22nd January 2024

- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. (a) Let $(i_1 i_2 \cdots i_k) \in S_n$ be a k-cycle, and $\sigma \in S_n$ an arbitrary permutation, prove that

$$\sigma(i_1 i_2 \cdots i_k)\sigma^{-1} = (\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)).$$

   (b) Suppose that $\sigma_i := (i_1 i_2 \cdots i_k)$ and $\sigma_j := (j_1 j_2 \cdots j_k)$ are k-cycles in S_n, prove that there exists some $\sigma \in S_n$ so that $\sigma_j = \sigma\sigma_i\sigma^{-1}$

   (c) Show that there are $\frac{n!}{(n-k)!k}$ many k-cycles in S_n.

2. Let σ be a k-cycle, prove that σ^i is also a k-cycle if and only if $\gcd(i,k) = 1$.

3. Let p be a prime number, prove that a permutation in S_n has order p if and only if it is a product of commuting p-cycles. Is the statement true if p is not assumed to be prime?

4. Let D_n be the n-th dihedral group, let $r \in D_n$ be a rotation so that $r^n = \mathrm{id}$, and let $s \in D_n$ be any reflection. It is known that $srs = r^{-1}$ (see optional Q6 of HW2). Suppose $n = 2k$ is even, prove that r^k commutes with every element in D_n.

5. Recall that every element in D_n can be expressed as either r^i for $0 \leq 1 < n$ or $r^j s$ for $0 \leq j < n$. Compute which element are $r^2sr^6sr^3$ and $sr^4sr^3sr^2$ in D_7 using the relation $sr = r^{-1}s$.

6. Explain why D_n and S_n are not cyclic.