THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 (2023-24, Term 2)
Honours Algebraic Structures
Homework 9
Due Date: 18th April 2024

Compulsory Part

1. Let F be a field, and let $a, b \in F$. Is $F[x]/(x-a)$ always isomorphic to $F[x]/(x-b)$? Justify your answer.

2. For each of the following pairs of rings, determine if they are isomorphic or not, and justify your answer:

   (a) $\mathbb{Z}_2[x]/(x^2+1)$ and $\mathbb{Z}_2[x]/(x^3+1)$.

   (b) $\mathbb{R}[x]/(x^2)$ and $\mathbb{R}[x]/(x^2-2x+1)$.

   (c) $\mathbb{Q}[x]/(x^2)$ and $\mathbb{Q}[x]/(x^2-1)$.

   (d) $\mathbb{R}[x]/(x^2+1)$ and $\mathbb{R}[x]/(x^2+2)$.

3. Let d be an integer which is not the square of an integer. Consider

$$\mathbb{Q}[\sqrt{d}] := \{a + b\sqrt{d} : a, b, \in \mathbb{Q}\} \subset \mathbb{C}.$$

   (a) Show that $\mathbb{Q}[\sqrt{d}]$ is a subring of $\mathbb{C}$, and hence an integral domain.

   (b) Define a **norm** $N : \mathbb{Q}[\sqrt{d}] \to \mathbb{Q}$ by $N(a + b\sqrt{d}) = a^2 - b^2 d$. Show that $N(\alpha\beta) = N(\alpha)N(\beta)$ for any $\alpha, \beta \in \mathbb{Q}[\sqrt{d}]$, and that $N(\alpha) = 0$ if and only if $\alpha = 0$.

   (c) Prove that $\mathbb{Q}[\sqrt{d}]$ is the smallest subfield of $\mathbb{C}$ containing both $\mathbb{Q}$ and $\sqrt{d}$.

   (d) Prove that $\mathbb{Q}[\sqrt{d}] \cong \mathbb{Q}[x]/(x^2 - d)$.

4. Determine if the following polynomials are irreducible in $\mathbb{Q}[x]$:

   (a) $f = x^3 + 6x^2 + 5x + 24257$
   (*Hint:* First consider f as an element in $\mathbb{Z}[x]$, then determine if its image $\bar{f}$ in $\mathbb{F}_2[x]$ is irreducible.)

   (b) $f = x^4 + x^2 + x + 1$

   (c) $f = 4x^3 - 6x - 1$

5. Determine if the following rings are fields. Justify your answers.

   (a) $\mathbb{Q}[x]/(x^{17} + 5x^2 - 10x + 45)$

   (b) $\mathbb{Z}[x]/(x^6 - 210x - 616)$. (Note: It is $\mathbb{Z}[x]$ instead of $\mathbb{Q}[x]$.)

   (c) $\mathbb{Q}[x]/(4x^3 - 6x - 1)$

   (d) $\mathbb{R}[x]/(x^{17} + 5x^2 - 10x + 45)$

Optional Part

1. Let F be a field. Let f, g be relatively prime polynomials in $F[x]$. Show that if both f and g divide a polynomial h in $F[x]$, then fg divides h.

2. Consider the polynomials $f = x^2 - x - 2$ and $g = x^3 - 2x + 1$ in $\mathbb{Z}_5[x]$. By adapting the Euclidean Algorithm to $\mathbb{Z}_5[x]$, find $a, b \in \mathbb{Z}_5[x]$ such that $af + bg = \gcd(f, g)$.

   (Here, $\gcd(f, g)$ is the unique monic polynomial in $\mathbb{Z}_5[x]$ with the property that the ideal (f, g) is equal to the principal ideal $(\gcd(f, g))$.)

3. Express the following polynomials as products of irreducible factors.

   (a) $x^4 + 1$ in $\mathbb{Z}_2[x]$.

   (b) $x^3 + 1$ in $\mathbb{Z}_3[x]$.

4. Show that the following polynomials are irreducible in $\mathbb{Q}[x]$:

   (a) $2x^5 + 25x + 210$

   (b) $17715x^2 + 1234567x + 4561$

   (c) $x^3 + 6x^2 + 7$

   (d) $4x^3 - 3x + \frac{1}{2}$

   (e) $\frac{1}{3}x^5 - x^4 + 1$

   (f) $x^4 + 5x^2 - 2x - 3$.

   (*Hint:* Consider the irreducible factors of the polynomial over $\mathbb{F}_2$ and $\mathbb{F}_3$. What conclusion can one draw?)

5. Let k be a field. Let $f = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial in $k[x]$ of degree n. Show that if f is irreducible in $k[x]$, then so is:

$$f^* := a_n + a_{n-1}x + \cdots + a_1 x^{n-1} + a_0 x^n.$$

6. Let F be a field, p a polynomial in $F[x]$. Then a theorem in our lecture notes says that the quotient ring $F[x]/(p)$ is a field if and only if p is irreducible in $F[x]$.

   Determine if each of the following rings is a field:

   (a) $\mathbb{Q}[x]/(x^3 - 1)$

   (b) $\mathbb{Q}[x]/(7x^{59} + 24x^9 + 6x + 156)$

   (c) $\mathbb{Q}[x]/(x^3 + x + 1)$

   (d) $\mathbb{Z}[x]/(x^3 + x + 1)$

   (e) $\mathbb{Q}/(17)$

   (f) $\mathbb{Z}/(17)$

   (g) $\mathbb{Z}[x]/(2, x)$

   (h) $\mathbb{Q}[x]/(x^2 - 3)$

   (i) $\mathbb{R}[x]/(x^2 - 3)$

(j) $\mathbb{R}[x]/(x^2+3)$

(k) $\mathbb{F}_5[x]/(x^2+1)$

(l) $\mathbb{R}[x]/(x^{17}+x^5+8x^2-x+1)$

7. **Converse of Euclid's Lemma.** Let F be a field, f a polynomial in $F[x]$ with degree ≥ 1, such that, for $g, h \in F[x]$, the condition $f \mid gh$ implies that $f \mid g$ or $f \mid h$. Show that f is irreducible in $F[x]$.

   (Try to prove this without invoking the unique factorization theorem.)