THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 (2023-24, Term 2)
Honours Algebraic Structures
Homework 8
Due Date: 4th April 2024

Compulsory Part

1. Let R be a ring which contains $\mathbb{C}$ as a subring. Show that there cannot be any ring homomorphism $R \to \mathbb{R}$.

2. Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \ldots$ be an increasing/ascending *chain* of ideals in a ring R. Show that the union $\bigcup_{i=1}^{\infty} I_i$ is an ideal in R.

3. Recall that in a commutative ring R, an element $a \in R$ is called **nilpotent** if $a^n = 0$ for some positive integer n, and the set N of all nilpotent elements is an ideal, called the **nilradical**, of R. Show that the quotient ring R/N has no nonzero nilpotent elements. (Such a ring is said to be **reduced**.)

4. Let R and R' be rings, and let I and I' be ideals of R and R' respectively. Let ϕ be a homomorphism of R into R'. Show that ϕ induces a natural ring homomorphism

$$\phi_* : R/I \to R'/I'$$

if $\phi(I) \subseteq I'$.

5. Let I be an ideal of a ring R, and let J be an ideal of R containing I. Show that J/I is an ideal of R/I, and there is a natural ring isomorphism

$$\frac{R/I}{J/I} \cong \frac{R}{J}.$$

6. Is $\mathbb{Z}[i]/(a + bi)$ always isomorphic to $\mathbb{Z}/(a^2 + b^2)$, for all $a, b \in \mathbb{Z}$? For example, is $\mathbb{Z}[i]/(2 + 2i)$ isomorphic to $\mathbb{Z}/8\mathbb{Z}$?

Hint: If $\mathbb{Z}[i]/(2+2i)$ is isomorphic to $\mathbb{Z}/8\mathbb{Z}$, then it is isomorphic to $\mathbb{Z}_8 = \{0, 1, 2, \ldots, 7\}$. Any isomorphism ϕ from $\mathbb{Z}/(2 + 2i)$ to $\mathbb{Z}_8$ must send 1 to 1, 0 to 0, and $\bar{i} = i + (2 + 2i)$ to some $a \in \mathbb{Z}_8$. What properties must this a satisfy? Does there exist $a \in \mathbb{Z}_8$ which satisfies all these properties?

Optional Part

1. Prove that the intersection of any set of ideals of a ring is an ideal.

2. Let n be a positive integer. Show that there cannot be a ring homomorphism from $\mathbb{Q}$ to $\mathbb{Z}_n$.

3. Let D be an integral domain, and let $a, b \in D$. Show that $(a) = (b)$ if and only if there exists a unit $u \in D^\times$ such that $a = ub$.

4. Let R be a commutative ring, and let u be a unit in R. Show that $R/(u)$ is isomorphic to the zero ring $\{0\}$.

5. (a) How many elements are there in $\mathbb{Z}_{12}/(3)$?

   (b) How many elements are there in $\mathbb{Z}_{12}/(5)$?

   (c) How many equivalence classes are there in $\mathbb{Z}_2[x]$ modulo the ideal generated by $x^3 + 1$? Give a representative in $\mathbb{Z}_2[x]$ for each of these equivalence classes.

6. Let a, b be integers. Show that $\mathbb{Z}[i]/(a + bi) \cong \mathbb{Z}[i]/(a - bi)$ by performing the following steps:

   (a) Define $\phi : \mathbb{Z}[i] \to \mathbb{Z}[i]/(a - bi)$ as follows:

   $$\phi(c + di) = \overline{c - di} := c - di + (a - bi), \quad c, d \in \mathbb{Z}.$$

   Show that ϕ is a ring homomorphism.

   (b) Show that ϕ is surjective.

   (c) Show that the kernel of ϕ is $(a + bi)$.

   (d) Apply the First Isomorphism Theorem for rings.

7. Let $R = C[-1, 1]$, the ring of continuous real-valued functions on $[-1, 1]$, equipped with the usual operations of addition and multiplication for real-valued functions. Let $I = \{f \in R : f(0) = 0\}$.

   (a) Show that I is an ideal in R.

   (b) Show that $R/I \cong \mathbb{R}$.

8. If D is a principal ideal domain and I is an ideal of D, prove that every ideal of the quotient D/I is principal.