THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 (2023-24, Term 2)
Honours Algebraic Structures
Homework 7
Due Date: 28th March 2024

Compulsory Part

1. Prove that if there exists a homomorphism from the zero ring to a ring R, then R must also be the zero ring.

2. Let m, n be relatively prime integers. Consider the map $\phi : \mathbb{Z}_{mn} \longrightarrow \mathbb{Z}_m \times \mathbb{Z}_n$:

$$\phi(a) := (a_m, a_n), \quad a \in \mathbb{Z}_{mn},$$

   where a_l denotes the remainder when a is divided by l.

   Show that ϕ is a ring isomorphism. (This is a version of the **Chinese Remainder Theorem**.)

3. Let R be a ring. The **center** $Z(R)$ of R is defined as

$$Z(R) := \{r \in R : rs = sr \text{ for all } s \in R\}.$$

   Show that $Z(R)$ is a subring of R.

4. Let R be a commutative ring. For $a \in R$, let:

$$I_a := \{x \in R : ax = 0\}$$

   Show that I_a is an ideal of R.

5. Let I, J be ideals of a commutative ring R. Show that the following are also ideals of R:
   (a) the intersection $I \cap J$,
   (b) the sum
$$I + J := \{r \in R : r = a + b \text{ for some } a \in I, b \in J\},$$
   and
   (c) the product
$$IJ := \left\{ r \in R : r = \sum_{i=1}^{n} a_i b_i \text{ for some } n \in \mathbb{N},\ a_i \in I,\ b_i \in J \right\}.$$

6. Recall that an element a of a ring R is called **nilpotent** if $a^n = 0$ for some positive integer n. Suppose that R is commutative. Show that the set N of all nilpotent elements in R is an ideal of R (called its **nilradical**).

Optional Part

1. Let R be the set of 2×2 real matrices of the form:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

   Given that R is a commutative ring under the usual operations of addition and multiplication for matrices, show that R is isomorphic to the field $\mathbb{C}$.

2. Determine if each of the following maps is a ring homomorphism:

   (a)
$$\phi : \mathbb{Z} \longrightarrow \mathbb{Q},$$
$$\phi(n) = n^2, \quad n \in \mathbb{Z}.$$

   (b)
$$\phi : \mathbb{Z}_6 \longrightarrow \mathbb{Z}_3,$$
$$\phi(s) = s_3, \quad s \in \mathbb{Z}_6,$$

   where s_3 denotes the remainder when s is divided by 3.

   (c)
$$\phi : \mathbb{Z}_7 \longrightarrow \mathbb{Z}/3\mathbb{Z},$$
$$\phi(s) = s + 3\mathbb{Z}, \quad s \in \mathbb{Z}_7,$$

   where $s + 3\mathbb{Z}$ denotes the residue of s (viewed as an integer) in the quotient ring $\mathbb{Z}/3\mathbb{Z}$.

3. Find a ring homomorphism from $\mathbb{Z}_7$ to $\mathbb{Z}_5$. If it does not exist, explain why not.

4. (a) Can there be a ring homomorphism from a non-integral domain to an integral domain? Why?

   (b) Can there be a ring homomorphism from an integral domain to a non-integral domain? Why?

5. Determine if the subset I is an ideal in the ring R, where:

   (a) $R = \mathbb{Z}[x]$ and I is the set of polynomials $\sum_{i=0}^{n} a_i x^i$ in $\mathbb{Z}[x]$ with the property that a_0 is odd.

   (b) $R = \mathbb{Z}[x]$ and I is the set of polynomials in $\mathbb{Z}[x]$ whose leading coefficients are even.

   (c) $R = \mathbb{Z}/6\mathbb{Z}$ and I is the set of elements $r + 6\mathbb{Z} \in \mathbb{Z}/6\mathbb{Z}$ such that r is an even number. (Note that I is well-defined, since if one representative of $r + 6\mathbb{Z}$ is even, so is any other element in the same congruence class modulo $6\mathbb{Z}$.)

6. Let m, n be nonzero integers. Show that if $\gcd(m, n) = 1$, then $(mn) = (m) \cap (n)$ in the ring $\mathbb{Z}$.