**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH 2078 (2023-24, Term 2)**
**Honours Algebraic Structures**
**Homework 6**
**Due Date: 21st March 2024**

## Compulsory Part

1. Find the units in the following rings:
   (a) $\mathbb{Z}$.
   (b) The ring $R$ of all real-valued functions on $\mathbb{R}$.
   (c) $D[x]$, where $D$ is an integral domain.

2. Show that the set $R^{\times}$ of units in a ring $R$ is a group under multiplication.

3. Let $R$ be a commutative ring. Show that the **binomial theorem** holds, i.e.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

   for any $a, b \in R$ and for any positive integer $n$.

4. An element $a$ of a ring $R$ is said to be **nilpotent** if $a^n = 0$ for some positive integer $n$. Show that if $a, b \in R$ are nilpotent and $ab = ba$, then $a + b$ is also nilpotent.

5. Let $D$ be an integral domain. If there exists a positive integer $n$ such that

$$na := \overbrace{a + \cdots + a}^{n \text{ times}} = 0$$

   for any $a \in D$, then $D$ is said to be of **finite characteristic**; in this case, we define the **characteristic** of $D$ to be

$$\text{char}(D) := \min\{n \in \mathbb{Z}_{>0} : na = 0 \ \forall a \in D\}.$$

   If no such positive integer exists, we say that $D$ is of **characteristic 0**, denoted as $\text{char}(D) = 0$.

   (a) Show that if $n1 \neq 0$ for any $n \in \mathbb{Z}_{>0}$, then $D$ is of characteristic 0; otherwise, we have
$$\text{char}(D) = \min\{n \in \mathbb{Z}_{>0} : n1 = 0\}.$$
   (b) Hence show that the characteristic of an integral domain is either 0 or a prime.

**Optional Part**

1. Show that $a^2 - b^2 = (a+b)(a-b)$ for all $a, b$ in a ring $R$ if and only if $R$ is commutative.

2. Let $R$ be a ring. If $a, b \in R$ are 0-divisors, is $a + b$ also a 0-divisor?

3. A ring $R$ such that $a^2 = a$ for any $a \in R$ is called a **Boolean ring**. Show that every Boolean ring is commutative.

4. Let $R$ be the set of all real-valued functions $f$ on $\mathbb{R}$ such that $f(0) = 0$. Let $+$ and $\cdot$ be the usual addition and multiplication operations for functions.

   (a) Show that $f + g \in R$ for all $f, g \in R$.

   (b) Show that $f \cdot g \in R$ for all $f, g \in R$.

   (c) With respect to $+$, what is the additive identity element of $R$, if it exists?

   (d) With respect to $\cdot$, what is the multiplicative identity element of $R$, if it exists?

5. (a) Is the product of two units in a ring necessarily a unit? If so, prove it; if not, provide a counterexample.

   (b) Is the sum of two units in a ring necessarily a unit? If so, prove it; if not, provide a counterexample.

6. Let $R$ be a nonzero commutative ring. Show that the polynomial ring $R[x]$ is an integral domain if and only if $R$ is an integral domain.

7. Let $D$ be an integral domain. Verify that under the convention that $\deg 0 := -\infty$, the following rules hold for all polynomials $f, g \in D[x]$:

   (a) $\deg(fg) = \deg f + \deg g$.

   (b) $\deg(f \pm g) \leq \max\{\deg f, \deg g\}$.

   Do we still have the above rules in $R[x]$ if the coefficient ring $R$ is no longer an integral domain?