THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 (2023-24, Term 2)
Honours Algebraic Structures
Homework 5
Due Date: 29th February 2024

Compulsory Part

1. Let $G = \{1, 2, 4, 5, 7, 8\}$. Define a binary operation $*$ on G as follows:

$$l * k = \overline{l \cdot k},$$

where $\cdot$ represents the multiplication of integers, and for any $n \in \mathbb{Z}$ the symbol $\overline{n}$ denotes the remainder (in $\mathbb{Z}_9 = \{0, 1, 2, \ldots, 8\}$) of the division of n by 9. Given that $G = (G, *)$ is group. Show that G is isomorphic to $\mathbb{Z}_6$.

2. Let $\phi : G \longrightarrow G'$ be a bijective group homomorphism. Show that the inverse map $\phi^{-1} : G' \longrightarrow G$ is also a group homomorphism.

3. Let:

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}.$$

   (a) Show that $(G, *)$ is a group, where $*$ is matrix multiplication.

   (b) Show that $(G, *)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

4. Determine if the following groups are isomorphic.

   (a) The additive group $(\mathbb{R}, +)$ and the multiplicative group $(\mathbb{R}_{>0}, \cdot)$.

   (b) The additive group $(\mathbb{Q}, +)$ and the multiplicative group $(\mathbb{Q}_{>0}, \cdot)$.

5. Determine if $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_6$. Justify your assertion.

6. Let G be a group.

   (a) Show that the map

$$\phi : G \longrightarrow G, \quad g \mapsto g^{-1}$$

   is a group homomorphism if and only if G is abelian.

   (b) Show that the map

$$\phi : G \longrightarrow G, \quad g \mapsto g^2$$

   is a group homomorphism if and only if G is abelian.

7. Let G be a group.

   (a) An isomorphism $\sigma : G \to G$ from G onto itself is called an **automorphism** of G. Show that the set $\mathrm{Aut}(G)$ of automorphisms of G forms a group under composition.

(b) Show that, for any $g \in G$, the map $i_g : G \to G$ defined by $i_g(a) = gag^{-1}$ for any $a \in G$ is an automorphism of G. The automorphism i_g is called an **inner automorphism** of G.

(c) Prove that the set $\mathrm{Inn}(G)$ of inner automorphisms of G is a normal subgroup of $\mathrm{Aut}(G)$.

Optional Part

1. Let $G = \{1, 5, 7, 11, 13, 17, 19, 23\}$. Define a binary operation $*$ on G as follows:

$$l * k = \overline{l \cdot k},$$

where $\cdot$ represents the multiplication of integers, and for any $n \in \mathbb{Z}$ the symbol $\overline{n}$ denotes the remainder of the division of n by 24.

   (a) Given that $G = (G, *)$ is group, show that G is *not* isomorphic to $\mathbb{Z}_8$.

   (b) G is isomorphic to one of the following groups. Make a guess which one.

      i. $S_2 \times \mathbb{Z}_4$.

      ii. $\mathbb{Z}_3 \times \mathbb{Z}_5$.

      iii. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

2. Let G, G' be isomorphic cyclic groups. Show that for any generator g of G (i.e. $G = \langle g \rangle$) and any group isomorphism $\phi : G \longrightarrow G'$, the element $\phi(g)$ is a generator of G'.

3. Show that any non-abelian group of order 6 is isomorphic to S_3.

4. Let n be a positive integer. Define $\phi : (\mathbb{Z}, +) \longrightarrow (\mathbb{Z}_n, +_n)$ as follows:

$$\phi(k) = \overline{k}, \quad k \in \mathbb{Z},$$

where $\overline{k}$ denotes the remainder of the division of k by n.

   (a) Show that ϕ is a group homomorphism.

   (b) Find $\ker \phi$ and the index $[\mathbb{Z} : \ker \phi]$.

   (c) Find all group homomorphism(s) $\psi : \mathbb{Z}_n \longrightarrow \mathbb{Z}$, if any exists.

5. Find the total number of group isomorphisms:

   (a) from U_5 to U_5.

   (b) from U_{12} to $\mathbb{Z}_{12}$.

6. Define a relation $\cong$ on groups as follows:

$$G \cong G' \quad \text{if } G \text{ is isomorphic to } G',$$

Show that $\cong$ is an equivalence relation.

7. (a) Let G be a group and $S \subset G$ be a generating set for G, i.e. we have $G = \langle S \rangle$. Let $\lambda : G \to G'$ and $\mu : G \to G'$ be two homomorphisms from G into a group G' such that $\lambda(s) = \mu(s)$ for any $s \in S$. Show that $\lambda = \mu$.

   (b) Use (a) to compute the order of $\mathrm{Aut}(\mathbb{Z}_{15})$. (More generally, what is the order of the automorphism group of a cyclic group of order n?)