THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 (2023-24, Term 2)
Honours Algebraic Structures
Homework 4
Due Date: 22nd February 2024

Compulsory Part

1. Write down all the cosets of the following subgroups

   (a) $\langle 4 \rangle \leq \mathbb{Z}$.

   (b) $\langle 4 \rangle \leq \mathbb{Z}_{12}$.

   (c) $\langle s \rangle \leq D_n$, where s is any reflection.

2. Find a cyclic subgroup of order 4 in S_4, and then give a list of its left coset representatives in S_4.

   (An element a in a group G is a called a **representative** of a left coset S of a subgroup H of G if $S = aH$. Note that a is a representative of S if and only if $a \in S$.)

3. Let G be a group of order pq, where p and q are (not necessarily distinct) prime numbers. Show that every proper subgroup of G is cyclic.

4. Let H be a subgroup of index 2 in a group G. Show that every left coset of H is also a right coset of H. Hence an index 2 subgroup must be normal.

5. Let G be a group and H, K be subgroups of G such that $K < H < G$. Suppose that $[G : H]$ and $[H : K]$ are finite. Show that $[G : K]$ is finite and we have

$$[G : K] = [G : H][H : K].$$

6. Let H and K be subgroups of finite index in a group G, and suppose that $[G : H] = m$ and $[G : K] = n$. Prove that $\text{lcm}(m, n) \leq [G : H \cap K] \leq mn$. Hence deduce that if m and n are relatively prime, then $[G : H \cap K] = [G : H][G : K]$.

Optional Part

1. Recall the definition of the **quaternion group**:

$$Q = \{\pm 1, \pm i, \pm j, \pm k\},$$

   where the group operation is written multiplicatively,

$$(-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1,$$

   the symbol 1 denotes the identity element, and -1 commutes with every element of the group.

   Consider the cyclic subgroup $H = \langle i \rangle$ of Q. Find $[Q : H]$, and give a list of representatives of the left cosets of H in Q.

2. Consider the dihedral group $D_6 = \{r_0, r_1, \ldots, r_5, s_1, s_2, \ldots, s_6\}$, where r_0 is the identity element, each r_k corresponds to the anticlockwise rotation by the angle of $2\pi k/6$, and the s_k's are reflections.

   (a) Find a subgroup of order 4 in D_6, if it exists.

   (b) Find a non-cyclic subgroup of order 6 in D_6, if it exists.

3. Prove that a group with at least 2 elements but containing no proper nontrivial subgroups must be cyclic and of prime order.

4. Let G be a group, and H be a subgroup of G such that its left cosets and right cosets give the same partition of G. Prove that H is normal in G.

5. Let G be a group and n be a positive integer. Let $H \leq G$ be the subgroup generated by all the order n elements in G. Prove that H is normal.