THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 (2023-24, Term 2)
Honours Algebraic Structures
Homework 3
Due Date: 8th February 2024

Compulsory Part

1. Determine whether the given subset is a subgroup (if it is, give a proof; if it is not, explain why):

   (a) The set $\mathbf{i}\mathbb{R}$ of all purely imaginary numbers inside $\mathbb{C}$.

   (b) The set $\{z \in \mathbb{C} : z^m = 1\}$ of m-th roots of unity inside the unit circle $U = \{z \in \mathbb{C} : |z| = 1\}$.

   (c) The set of $n \times n$ matrices with determinant -1 inside GL$(n, \mathbb{R})$.

   (d) The set of $n \times n$ matrices M such that $M^T M = I$, where M^T denotes the transpose of M and I is the $n \times n$ identity matrix, inside GL$(n, \mathbb{R})$.

2. Consider the cyclic group $\mathbb{Z}_{20}$.

   (a) Write down all the generators of $\mathbb{Z}_{20}$.

   (b) List all the subgroups of $\mathbb{Z}_{20}$, and for each subgroup, compute its order and write down all its generators.

3. Let G be a group. Show that a *finite* nonempty subset H of G is a subgroup of G if and only if it is closed under the group operation of G (i.e. $ab \in H$ for all $a, b \in H$).

4. Let H and K be subgroups of an abelian group G. Show that

   $$\{hk : h \in H \text{ and } k \in K\}$$

   is also a subgroup of G.

   Give an example to show that this is not the case when G is nonabelian.

5. In the lecture notes, we defined the subgroup generated by a nonempty subset S in a group G as the set

   $$\langle S \rangle := \{a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} : n \in \mathbb{N}, a_i \in S, m_i \in \mathbb{Z}\}.$$

   Prove rigorously that $\langle S \rangle$ is the intersection of all subgroups in G containing H, i.e.

   $$\langle S \rangle = \bigcap_{\{H :\, S \subset H < G\}} H.$$

6. Let G be an abelian group. Show that the set H consisting of those elements of G which have finite orders is a subgroup of G.

Optional Part

1. Determine whether the given subset is a subgroup (if it is, give a proof; if it is not, explain why):
   (a) The set $e\mathbb{Q}$ of rational multiples of the number e inside $\mathbb{R}$.
   (b) The set $\{\pi^n : n \in \mathbb{Z}\}$ inside $\mathbb{R}$.
   (c) The set of diagonal $n \times n$ matrices with no zeros on the diagonal inside $\mathrm{GL}(n, \mathbb{R})$.
   (d) The set of $n \times n$ matrices with determinant ± 1 inside $\mathrm{GL}(n, \mathbb{R})$.
2. Express each element in S_3 as a product of powers of (123) and (12) (e.g. $(23) = (123)^2(12)$), if possible.
3. In S_6, how many subgroups are of
   - order 5?
   - order 3?
4. Find a non-cyclic subgroup of order 4 in S_4, if it exists. If it does not exist, explain why not.
5. Let n be an integer larger than or equal to 4. Let r be the anticlockwise rotation by $2\pi/n$ in the dihedral group D_n. Let s be a fixed reflection in D_n. Find the order of the subgroup $H = \langle r^2, s\rangle$ in D_n if:
   (a) n is odd.
   (b) n is even.
6. Show that a group with infinitely many elements has infinitely many subgroups.