THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 (2023-24, Term 2)
Honours Algebraic Structures
Homework 2
Due Date: 1st February 2024

Compulsory Part

1. Let $\omega = e^{\pi i/12} \in \mathbb{C}$. Find the order of ω^8, ω^{13}, ω^{22} and ω^{2078} respectively as elements in the multiplicative group $\mathbb{C}^\times$ of the complex number.

2. Consider the special linear group $\mathrm{SL}(2, \mathbb{R})$.

   (a) Find an element of order 2 in $\mathrm{SL}(2, \mathbb{R})$.

   (b) Find an element of order 3 in $\mathrm{SL}(2, \mathbb{R})$.

   (c) Find an element of infinite order in $\mathrm{SL}(2, \mathbb{R})$.

3. Let G be a group. Show that for all $a, b \in G$ such that $|ab|$ is finite, we have $|ab| = |ba|$.

4. Let $\mu_1, \mu_2 \in S_n$ be two disjoint cycles. Using the fact that disjoint cycles commute, show that
$$|\mu_1\mu_2| = \mathrm{lcm}(|\mu_1|, |\mu_2|).$$
More generally, show by induction that if $\mu_1, \mu_2, \ldots, \mu_r \in S_n$ are disjoint cycles, then
$$|\mu_1\mu_2 \cdots \mu_r| = \mathrm{lcm}(k_1, k_2, \ldots, k_r),$$
where $k_i = |\mu_i|$ for $i = 1, 2, \ldots, r$.

5. How many elements are of order 2 in D_6?

6. Let G be a finite group with an even number of elements. Show that there must be an order 2 element $a \in G$.

Optional Part

1. Let a, b be elements of a group G. Suppose a has order 5 and $a^3b = ba^3$. Prove that $ab = ba$.

2. By definition, the **orthogonal group** $\mathrm{O}(2, \mathbb{R})$ consists of real 2×2 matrices A which satisfy the condition
$$A^tA = AA^t = I,$$
where A^t denotes the transpose of A, and I is the 2×2 identity matrix.

   (a) Show that $\mathrm{O}(2, \mathbb{R})$ is a group under matrix multiplication.

   (b) Find an element of order 2 in $\mathrm{O}(2, \mathbb{R})$.

   (c) Find an element of order 3 in $\mathrm{O}(2, \mathbb{R})$.

3. Find the order of the following elements in S_7:
   (a) (1325).
   (b) $(1325)(47)$.
   (c) $(1325)(647)$.
   (d) $(35)(46)(37)(32)$.

4. Consider the permutations

$$\sigma = (1264)(2513),\ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 7 & 5 & 1 & 3 & 6 \end{pmatrix} \in S_7.$$

   (a) Express σ and τ as
      i. a product of transpositions, and
      ii. a product of disjoint cycles.
   (b) Compute $|\sigma|$, $|\tau|$ and $|\sigma\tau|$.

5. (a) How many elements are of order 3 in S_5?
   (b) How many elements are of order 4 in S_6?
   (c) How many elements are of order 3 in S_7?

6. Consider the dihedral group D_n, where n is an integer greater than 2. For $j = 0, 1, 2, \ldots, n-1$, let r_j denote the anticlockwise rotation about the origin by $2\pi j/n$.
   (a) Show that for any $k \in \mathbb{N}$, rotation $r \in D_n$ and reflection $s \in D_n$, we have

$$sr^k s = (srs)^k.$$

   (b) Show that for any reflection $s \in D_n$ and any rotation $r \in D_n$, we have

$$srs = r^{-1}.$$

   (*Hint*: First prove the identities for one particular reflection s, then use the fact that any other reflection s' is equal to sr' for some rotation $r' \in D_n$.)