**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH 2078 (2023-24, Term 2)**
**Honours Algebraic Structures**
**Homework 1**
**Due Date: 18th January 2024**

## Compulsory Part

1. Let
$$T := \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{C},\, xy = 1 \right\}.$$
Show that $T$ is a group under matrix multiplication.

2. A map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is called an **affine linear transformation** if there exist $A \in \mathrm{GL}(n, \mathbb{R})$ and $b \in \mathbb{R}^n$ such that $\varphi(x) = Ax + b$ for all $x \in \mathbb{R}^n$. Show that the set $\mathrm{Aff}(n, \mathbb{R})$ of affine linear transformations on $\mathbb{R}^n$ forms a group under composition of maps. This group models $n$-dimensional *real affine geometry*.

3. Prove Proposition 1.1.7 in the lecture notes: Let $G$ be a group.

   (a) For all $g \in G$, we have:
   $$(g^{-1})^{-1} = g.$$

   (b) For all $a, b \in G$, we have:
   $$(ab)^{-1} = b^{-1}a^{-1}.$$

   (c) For all $g \in G$, $n, m \in \mathbb{Z}$, we have:
   $$g^n \cdot g^m = g^{n+m}.$$

   Write down all your arguments in a rigorous way.

4. Let $G_1, G_2$ be groups. Show that the Cartesian product $G_1 \times G_2$ is a group under the operation
$$(a_1, b_1) * (a_2, b_2) := (a_1 *_1 a_2, b_1 *_2 b_2)$$
for $a_1, a_2 \in G_1$ and $b_1, b_2 \in G_2$, where $*_1, *_2$ are the group operations of $G_1, G_2$ respectively. The group $G_1 \times G_2$ is called the **direct product** of $G_1$ and $G_2$.

   Can you also define the direct product of an arbitrary collection $\{G_i : i \in I\}$ of groups (here $I$ is an arbitrary, possibly infinite and even uncountable, index set)?

5. Let $G$ be a group. Show that the equation
$$x^2 = x$$
has a *unique* solution in $G$.

**Optional Part**

1. Determine whether the given set equipped with the given binary operation is a group (if it is, give a proof; if it is not, explain why):
   (a) The set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers, equipped with addition.
   (b) The set $\mathbb{R}_{>0}$ of positive real numbers, equipped with multiplication.
   (c) The set $2\mathbb{Z} = \{\dots, -4, -2, 0, 2, 4, \dots\}$ of even integers, equipped with addition. (How about odd integers?)
   (d) The set $U := \{z \in \mathbb{C} : |z| = 1\}$ of complex numbers with modulus 1, equipped with multiplication.
   (e) The set $\{z \in \mathbb{C} : \operatorname{Im} z = 1\}$ of complex numbers with imaginary part equals to 1, equipped with multiplication.
   (f) The set $M_{m\times n}(\mathbb{C})$ of $m \times n$ complex matrices, equipped with matrix addition.
   (g) The set of $2 \times 2$ matrices with integer coefficients whose determinants are non-zero, equipped with matrix multiplication.
   (h) The set $H = \mathbb{R} \times \mathbb{R}$, equipped with the operation $*$ defined by

   $$(x_1, y_1) * (x_2, y_2) = (x_1 + x_2, y_1 + y_2 + x_1 x_2)$$

   for $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

2. Let

   $$R = \{r \in \mathbb{Q} : \text{ there exist } n \in \mathbb{Z}_{>0} \text{ such that } 2^n r \in \mathbb{Z}\}.$$

   Is $R$ a group under addition? Justify your answer.

3. The **quaternion group** is defined as follows:

   $$Q = \{1, -1, i, j, k, -i, -j, -k\},$$

   where the group operation is written multiplicatively, the symbol 1 denotes the identity element, and $-i, -j, -k$ denotes $(-1)i$, $(-1)j$, $(-1)k$, respectively.

   Moreover, by definition $-1$ commutes with every element of the group (for instance, $(-1)i = i(-1) = -i$), and the symbols $i, j, k$ satisfy the following relations:

   $$(-1)^2 = 1, \quad i^2 = j^2 = k^2 = ijk = -1.$$

   (a) Show that $ij = k$ and $jk = i$.
   (b) Show that $ij = -ji$.

4. If $a, b \in G$ are **commuting** elements in a group $G$, i.e. $ab = ba$, show that $(ab)^n = a^n b^n$ for any $n \in \mathbb{Z}$.

5. Prove rigorously that the set $\mathrm{Aut}_{\mathcal{C}}(A)$ of automorphisms of an object $A$ in a category $\mathcal{C}$ is a group with identity $\mathbf{1}_A$.