MATH2068 Honour Mathematical Analysis II

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Clive Chan

Figure 1: Maybe Math-addiction is not curable?

We went through the following:

Theorem 1 (Integral Mean value theorem)**.** *Let f integrable and g continuous, non-negative on $[a,b]$, then*

$$\int_a^b f(x)g(x)\mathrm{d}x = f(c)\int_a^b g(x)\mathrm{d}x$$

for some $c \in (a,b)$.

Remarks. An intuition for the formula: we know that

$$F(b) - F(a) = F'(c)(b-a)$$

for F differentiable on (a,b) and continuous on $[a,b]$.

Put $F(x) = \int_a^x f(x)\mathrm{d}x$, then the above equation becomes

$$\int_a^b f(x)\mathrm{d}x = f(c)(b-a) = f(c)\int_a^b \mathrm{d}x.$$

Note that $J \mapsto \int_J g(x)\mathrm{d}x$ is a measure on the subintervals J because g is non-negative, so one may naturally expect the intergral mean value theorem to hold because it is just changing the measure from $\mathrm{d}x$ to $\mathrm{d}\mu$, $\mu(J) = \int_J g$.

Example 1. Show that $\lim_{n\to\infty}\int_0^{\frac{\pi}{2}} \cos^n x\mathrm{d}x = 0$.

Proof. $\cos^n x = \cos x \cdot \cos^{n-1} x$, both are differentiable on $(0, \frac{\pi}{2})$ and continuous on $[0, \frac{\pi}{2}]$. Moreover, they are both non-negative on $[0, \frac{\pi}{2}]$.

Note that $c \in (0, \frac{\pi}{2}$ implies $\cos c < 1$, so $\cos^n c \to 0$. Hence,

$$\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \cos^n x \mathrm{d}x = \lim_{n\to\infty} \cos^{n-1} c \int_0^{\frac{\pi}{2}} \cos x \mathrm{d}x = 0$$

because we know that the definite integral is finite. □

Example 2. Show that $\lim_{n\to\infty} \int_0^1 \frac{x^n}{x^2+1} \mathrm{d}x = 0$.

Proof. Similarly, after checking all the assumptions, we note that $\lim_{n\to\infty} c^{n-1} = 0$ for any $0 < c < 1$ and $\int_0^1 \frac{x}{x^2+1} \mathrm{d}x < \infty$. □

Next we prove a mean value type result in two propositions.

Proposition 2. *If $f : [a, b] \to \mathbb{R}$ is Riemann-integrable on $[a, b]$ then there exists at least one point $c \in (a, b)$ at which f is continuous.*

Proof. By integrability for all $\epsilon > 0$ there exists a partition $P = \{x_0, \cdots, x_n\}$ such that

$$\epsilon(b - a) > U(f, P) - L(f, P) = \sum_{k=1}^{n} osc(f, [x_{k-1}, x_k])(x_k - x_{k-1}).$$

Assume $osc(f, [x_{j-1}, x_j]) \geq \epsilon$ for all j then the RHS above becomes $\geq \epsilon \sum(x_k - x_{k-1}) = \epsilon(b - a)$, which is a contradiction.

Hence there exists some j such that $osc(f, [x_{j-1}, x_j]) < \epsilon$. Any sub-interval of $[x_{j-1}, x_j]$ has no greater oscillation and we can pick instead a subinterval of length $< \epsilon$ if $[x_{j-1}, x_j]$ is not shorter than ϵ. In this way we see that there exists a subinterval $J \subset (a, b)$ such that both the oscillation on J and the length of J are less than ϵ.

Pick $\epsilon = 1$ and denote by J_1 what we obtained above. Repeat the argument on J_1 with $\epsilon = \frac{1}{2}$ to obtain a J_2. Inductively obtain a nested sequence $\{J_n\}$. By nested interval theorem there exists a point $c \in \cap_n J_n$.

For any ϵ, there exists $n \in \mathbb{N}$ such that if $x \in J_n$ then $|f(x) - f(c)| < \frac{1}{n} < \epsilon$. Note that $|x - c| < \frac{1}{n}$ by definition of J_n. Hence f is continuous at c. □

Proposition 3. *If $f : [a,b] \to \mathbb{R}$ is Riemann-integrable on $[a,b]$ then for every $\epsilon > 0$ there exists $c_i \in (a,b)$ $(i = 1,2)$ such that f is continuous at c_i $(i = 1,2)$ and*

$$\begin{cases} f(c_1)(b-a) & < \int_a^b f(x)\mathrm{d}x + \epsilon \\ \int_a^b f(x)\mathrm{d}x - \epsilon & < f(c_2)(b-a) \end{cases}$$

Proof. Fix $\epsilon > 0$, by integrability there exists $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a,b]$ such that

$$U(f,P) = \sum_{k=1}^{n} \sup_{x_{k-1} \le x \le x_k} f(x)(x_k - x_{k-1}) < \int_a^b f(x)\mathrm{d}x + \epsilon.$$

Similar as in the last preposition, there exists j such that

$$\sup_{x_{j-1} \le x \le x_j} f(x) < \frac{1}{b-a}\left(\int_a^b f(x)\mathrm{d}x + \epsilon\right).$$

Apply the previous proposition on any non-degenerate (length non-zero) subinterval, we see that f is continuous in a dense subset of $[a,b]$, so there exists a point $c \in [x_{j-1}, x_j]$ such that f is continuous at c. Now $f(c) < \sup_{x_{j-1} \le x \le x_j} f(x) < \frac{1}{b-a}\left(\int_a^b f(x)\mathrm{d}x + \epsilon\right)$. $\square$

Refer to arXiv: 1106.1807, "Mean Value Integral Inequalities" by Rodrigo López Pouso.