# MATH2068 Honour Mathematical Analysis II

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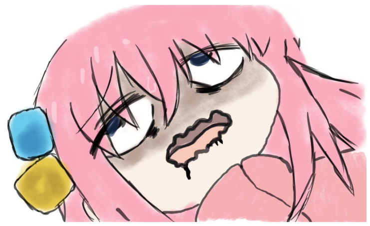

Figure 1: Maybe students' faces look the same after Maths tutorial?

Today we demonstrate some examples of Riemannian integrable functions.

In the definition of Riemannian integrability, we need to prove something true for all partition on a given $[a,b]$, but it is impossible to consider all partitions on $[a,b]$. What should we do then?

We use the fact that

$$L(f,P_n) \le \underline{\int_a^b} f(x)\mathrm{d}x \le \overline{\int_a^b} f(x)\mathrm{d}x \le U(f,P_n)$$

where $P_n$ is a partition on $[a,b]$ for each $n$, and try to find a sequence $(P_n)$ such that (denote the limit by $L$ for convenience)

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n)(=L).$$

By squeeze theorem this forces

$$\underline{\int_a^b} f(x)\mathrm{d}x = \overline{\int_a^b} f(x)\mathrm{d}x$$

and therefore $f$ is integrable on $[a,b]$.

*Remarks.* In test you are expected to explain why the above argument works before you apply it. Well, as you are already in the second year, you should know that the contents in tutorials cannot be taken for granted in tests.

Self-study Question: Can we claim that the integral equals $L$? (This is not an open-ended question!)

**Example 1.** $f(x) = x^2$ is integrable on $[a,b]$ for any $a \ne b, a, b \in \mathbb{R}$.

*Proof.* Choose a sequence of partition depending on $n$, defined by

$$P_n = \{0, \frac{b-a}{n}, \cdots, \frac{b-a}{n}k, \cdots, b-a\},$$

then

$$\begin{aligned}
U(f, P_n) &= \sum_{k=1}^{n} x_k^2 \frac{b-a}{n} \\
&= \sum_{k=1}^{n} \left(\frac{b-a}{n}k\right)^2 \frac{b-a}{n} \\
&= \frac{(b-a)^3}{n^3} \sum_{k=1}^{n} k^2 \\
&= \frac{(b-a)^3}{n^3} \frac{n(n+1)(2n+1)}{6} \\
&= \frac{1}{6}(b-a)^3(1+\frac{1}{n})(2+\frac{1}{n}) \\
&\xrightarrow{n\to\infty} \frac{1}{3}(b-a)^3.
\end{aligned}$$

Similarly, for the lower sum we have

$$\begin{aligned}
L(f, P_n) &= \sum_{k=0}^{n-1} x_k^2 \frac{b-a}{n} \\
&= \sum_{k=0}^{n-1} \left(\frac{b-a}{n}k\right)^2 \frac{b-a}{n} \\
&= \frac{(b-a)^3}{n^3} \sum_{k=0}^{n-1} k^2 \\
&= \frac{(b-a)^3}{n^3} \frac{(n-1)n(2n-1)}{6} \\
&= \frac{1}{6}(b-a)^3(1-\frac{1}{n})1(2-\frac{1}{n}) \\
&\xrightarrow{n\to\infty} \frac{1}{3}(b-a)^3 \\
&= \lim_{n\to\infty} U(f, P_n)
\end{aligned}$$

By the argument at the beginning of today, the function $f(x) = x^2$ is integrable on $[a, b]$. □

**Example 2.** $f(x) = \sin x$ is integrable over $x \in [0, \pi]$.

*Proof.* Another method to prove integrability is, show that for any $\epsilon > 0$ there exists a partiton $P$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

Equivalently (due to Archimedean principle), we can show that for any $n \in \mathbb{N}$ there exists a partition $P_n$ such that

$$U(f, P_n) - L(f, P_n) < \frac{1}{n},$$

but this is not always the most convenient idea. Let us follow from the most fundamental definition here.

Before we do anything, it is possible to expect that the estimation will be quite similar to the continuity of $f$, and intuitively we may believe that the following argument is natural when we are integrating on compact intervals $[a, b]$. If you don't see this immediately, let's read the following argument first:)

Define $P_n = \{0, \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \cdots, \pi\}$. Then for any $\epsilon > 0$, by the uniform continuity of the sine function on $[0, \pi]$, there exists $n > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{\pi}$ whenever $|x - y| < \frac{\pi}{n}$.

$$\begin{aligned}
U(f, P) - L(f, P) &= \frac{\pi}{n} \sum_{k=0}^{n} \left( \sup_{P_k} f - \inf_{P_k} f \right) \\
&< \frac{\pi}{n} (n \frac{\epsilon}{\pi}) \\
&= \epsilon.
\end{aligned}$$

□

*Remarks.* Continuity on a compact interval implies uniform continuity, so we are not using something too specific. If you find this too surprising, this is a good alarm for you to start revising MATH2050/8.

Extra question (it's open-ended to me, but seems like it is a classical problem, and is solved many decades ago): in measure theory we often break down functions into "layer-cakes", for instance, if $u \geq 0$ is $L^1$, then

$$\int_{\Omega} u d\mu = \int_0^{\infty} \mu\{u \geq t\} \mathrm{d}t$$

where $\Omega$ is open. Do you think there is an analogous result in Riemann integration theory?