MATH2068 Honour Mathematical Analysis II

Week 5, 5 Feb 2024

Clive Chan

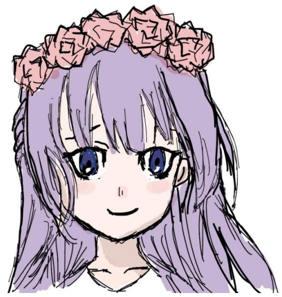

Figure 1: This week we have little Shoko:)

Claim 1. *If f is C^1 on (a, b), then f is Lipschitz on any closed and bounded subinterval $[c, d] \subset (c, b)$.*

Proof. By C^1, f' is continuous on (a, b) hence on $[c, d]$, $[c, d]$ compact so f' is bounded on $[c, d]$, i.e. there exists M such that $|f'| \leq M$ on $[c, d]$. For any $x, y \in [c, d]$, we have

$$|f(x) - f(y)| = |f'(k)||x - y|$$

for some $k \in (x, y)$ by the mean value, and the RHS is $\leq M|x - y|$. Hence,

$$|f(x) - f(y) \leq M|x - y|$$

for all $x, y \in [c, d]$ and f is M-Lipschitz on $[c, d]$. □

Claim 2. *Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function, then there exists a sequence of step functions (φ_n) on $[a, b]$ such that*

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \int_a^b \varphi_n(x)\mathrm{d}x.$$

Proof. By integrability,

$$\int_a^b f = \overline{\int_a^b} f = \inf U(f, P).$$

For any $n \in \mathbb{N}$, there exists a partition P_n such that

$$\int_a^b f + \frac{1}{n} > U(f, P_n)$$

Denote $(I_{n,i})_i$ be the subintervals defined by P_n and $M_i = \sup_{I_{n,i}} f$, then

$$\int_a^b \sum_i M_i \chi_{I_{n,i}} = U(f, P_n).$$

Let $s_n(x) = \sum_i M_i \chi_{I_{n,i}}$, we have

$$\left| \int_a^b f - s_n \right| < \frac{1}{n}$$

for all n and hence (s_n) is a desired sequence of step functions. □

Claim 3. *Using the same setting for f as above, find a sequence of continuous functions (g_n) on $[a, b]$ such that we have*

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \int_a^b g_n(x)\mathrm{d}x.$$

Proof. Define $P_n, (I_{n,i})$ in the same way as above. Suppose $I_{n,i} = [x_{n,i-1}, x_{n,i}]$, then $\frac{f(x_{n,i} - f(x_{n,i-1}}{x_{n,i} - x_{n,i-1}} t + f(x_{n,i-1}), \exists t \in [0,1]$ (any point along the y-axis between $f(x_{n,i-1})$ and $f(x_{n,i}))$ is less then either $f(x_{n,i})$ or $f(x_{n,i-1})$, hence $\leq M_i$ because $f(x_{n,i}), f(x_{n,i-1}) \in I_{n,i}$.

Define $g_n(x)$ to be the piecewise linear function by joining $(x_{n,i})$, i.e., define

$$g_n(x) = (f(x_{n,i}) - f(x_{n,i-1}))t + f(x_{n,i-1}) \text{ if } x = x_{n,i-1} + (x_{n,i} - x_{n,i-1})t \text{ for some } i \text{ and } t \in [0,1)$$

When $x \notin P_n$ there exists unique i such that $x \in I_{n,i}$ and hence unique t to define $g_n(x)$. When $x \in P_n$ we check that $f(x_{n,i} = f(x_{n,i}) + (f(x_{n,i+1}) - f(x_{n,i}))0 = f(x_{n,i-1}) + (f(x_{n,i}) - f(x_{n,i-1})1$ so g_n is well-defined at x.

Also, g_n is continuous (check!).

Now we estimate:

$$\begin{aligned}
\left| \int_a^b f - g_n \right| &\leq \int_a^b |f - g_n| \\
&\leq \sum_i \int_{x_{n,i-1}}^{x_{n,i}} M_i - f \\
&= U(f, P_n) - \int_a^b f \\
&< \frac{1}{n}.
\end{aligned}$$

Hence, (g_n) is a desired sequence of continuous functions. □

We recall a theorem here:

Theorem 4 (Darboux; Bartle 6.2.12). *If f is differentiable on $I = [a, b]$, and if k is a number between $f'(a)$ and $f'(b)$, then there is at least one point $c \in (a, b)$ such that $f'(c) = k$.*

Claim 5. *Let f be differentiable on (a, b). If there exists $c \in (a, b)$, such that $f'(c) > 0$, and for any $x \in (a, b)$ we have $f'(x) \neq 0$, then $f'(x) > 0$ for all $x \in (a, b)$.*

Proof. We may directly use the theorem: if $f'(x) \leq 0$ for some x, then by assumption we have $f'(x) < 0$ and therefore some point k between x, c has derivative 0.

We may also use the proof of Darboux theorem: if $f'(x) < 0$ for some x, then on the closed subinterval $[c, x]$ (or $[x, c]$; we may assume $c < x$), since f has no maximum at x, c, f must have interior maximum at some $k \in [c, x]$ (because f must attain maximum on a compact domain), which makes $f'(k) = 0$, contradiction arises. ☐