MATH2068 Honour Mathematical Analysis II

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Figure 1: This week we have Kana (Piman ver.), also drew by my friend:)

Theorem 1 (Warm up)**.** *Let $-\infty < a < b < \infty$ and let f, g be differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose that*

$$\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x).$$

If $\displaystyle\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{\pm\infty\}$, then $\displaystyle\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

Proof. CASE I: Suppose $L < \infty$.

By the definition of right limit, for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for all $x \in (a, a + \delta)$ we have

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon.$$

For any $(x, y) \subset (a, a + \delta)$, the Cauchy Mean Value Theorem tells us that there exists $u \in (x, y)$ such that

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(u)}{g'(u)}.$$

Since $u \in (x, y) \subset (a, a + \delta)$, we have

$$\left| \frac{f'(u)}{g'(u)} - L \right| < \epsilon$$

which is equivalent to

$$\left|\frac{f(y)-f(x)}{g(y)-g(x)}-L\right|<\epsilon.$$

Take limit $x \to a^+$, $f(x), g(x) \to 0$ (because what we have obtained is true for all $x \in (a, a+\delta)$) so we obtain

$$\left|\frac{f(y)}{g(y)}-L\right| \leq \epsilon$$

and this inequality is true for all $y \in (a, a+\delta)$. Hence the right limit of $\frac{f(y)}{g(y)}$ is L.

CASE II: Suppose $L = \infty$.

By the definition of right limit, for any $M > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for all $x \in (a, a+\delta)$ we have

$$\frac{f'(x)}{g'(x)} > M$$

For any $(x, y) \subset (a, a+\delta)$, the Cauchy Mean Value Theorem tells us that there exists $u \in (x, y)$ such that

$$\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f'(u)}{g'(u)}.$$

Since $u \in (x, y) \subset (a, a+\delta)$, we have

$$\frac{f'(u)}{g'(u)} > M$$

which is equivalent to

$$\frac{f(y)-f(x)}{g(y)-g(x)} > M.$$

Take limit on x, and then y as above. □

Theorem 2. *Suppose* $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f(x) = 0$ *and* $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$, *then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

Proof. We start with the case $\infty > L$.

Similar to the above argument, for any $\epsilon > 0$ there exists K such that for all $x, y > K$ we have

$$L - \epsilon < \frac{f(x)-f(y)}{g(x)-g(y)} < L + \epsilon$$

Take limit $y \to \infty$ we obtain

$$L - \epsilon \leq \frac{f(x)}{g(x)} \leq L + \epsilon$$

and the result follows.

When $L = \infty$, our argument would be starting with: for any $M > 0$ there exists K such that for all $x, y > K$ we have

$$\frac{f(x) - f(y)}{g(x) - g(y)} > M$$

Take limit on y then

$$\frac{f(x)}{g(x)} > M$$

for all $x > K$. The result follows. $\square$

Theorem 3. *Suppose* $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f(x) = \infty$ *and* $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$*, then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

Proof. We start with the case $\infty > L > 0$.

By $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$ we know that for all $\epsilon > 0$ there exists $K_1 = K_1(\epsilon) > 0$ such that for all $x > K_1$ we have

$$L - \epsilon < \frac{f'(x)}{g'(x)} < L + \epsilon$$

Pick any $x, y > K_1$, by Cauchy mean value theorem, there exists $u \in (x, y)$ such that

$$\frac{f'(u)}{g'(u)} = \frac{f(y) - f(x)}{g(y) - g(x)}.$$

Since $u \in (x, y)$ implies $u > K_1$, we have $L - \epsilon < \frac{f(y)-f(x)}{g(y)-g(x)} < L + \epsilon$.

Since $g \to \infty$ we can assume $g(x), g(y) > 0$. Or, to be clumsy, we can find K_1' such that $g(x) > 0$ whenever $x > K_1'$ and replace K_1 by $\max\{K_1, K_1'\}$.

Fix x, for any $d > 0$ to be determined, there exists K_2 such that for all $y > K_2$ we have $0 < \frac{g(x)}{g(y)} < d$ (note: the fraction is positive), which gives

$$(L - \epsilon)\frac{g(y) - g(x)}{g(y)} < \frac{f(y) - f(x)}{g(y) - g(x)}\frac{g(y) - g(x)}{g(y)} < (L + \epsilon)\frac{g(y) - g(x)}{g(y)}$$

$$(L - \epsilon)(1 - d) < \frac{f(y) - f(x)}{g(y)} < (L + \epsilon)(1 - d) < L + \epsilon$$

For the same d we can also find K_3 such that for all $y > K_3$ we have $-d < \frac{f(x)}{g(y)} < d$, hence for any $y > K_3$ we have

$$(L - \epsilon)(1 - d) - d < \frac{f(y)}{g(y)} < L + \epsilon + d$$

Choose $d = \min\{\frac{\epsilon}{L-\epsilon}, \epsilon\}$, this gives $(L-\epsilon)(1-d) \geq L - 2\epsilon$ and $d \leq \epsilon$. As d is fixed, we may fix K_2, K_3, let $K = \max\{K_1, K_2, K_3\}$, then for all $x, y > K$ we have

$$L - 3\epsilon < \frac{f(y)}{g(y)} < L + 2\epsilon$$

This implies $\lim_{y\to\infty} \frac{f(y)}{g(y)} = L$. □

I think students used l'Hopital's rule a lot in MATH1018, just without a rigorous treatment on the proof behind.