MATH2068 Honour Mathematical Analysis II

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Figure 1: It seems like many of you like Anya, and my friend drew one for you

Basically, the first part of the course is a reformulation on HKDSE mathematics, using the rigorous language. For instance, we proved the following relationship for a function f defined on an open interval $I \ni c$:

$$c\text{: interior max. of } f \;\overset{f \in C^2(I) \,\&\, f''(c) < 0}{\longleftarrow}\; f'(c) = 0$$

$$c\text{: interior max. of } f \;\longrightarrow\; f'(c) = 0$$

The proof of the right arrow is using Caratheodory's result, which gave a φ continuous at c and $\varphi(c) = f'(c)$. The proof of the left arrow is, roughly speaking, by Taylor (with remainder $\frac{1}{2}f''(x_0)(x - x_0)^2, x_0 \in (c - r, c + r),\ f''(x_0) > 0$) we know that f is locally a parabola opens upward, and then we make use of the continuity of f'' at x_0.

Today we look at l'Hopital's theorem and a generalization of it.

Theorem 1 (l'Hospital)**.** *Let f, g be differentiable on $(a, b) \ni c$, $f(c) = g(c) = 0$ and for points other than c we have $g, g' \neq 0$. Then $\lim_{x\to c} \frac{f}{g} = \lim_{x\to c} \frac{f'}{g'}$ if the RHS exists.*

Remarks. If RHS DNE, the l'Hopital theorem does not say that LHS DNE.

Theorem 2 (Stolz-Cesaro)**.** *Suppose $(b_n) > 0 \subset \mathbb{R}, \sum b_n \to \infty$, then for all $(a_n) \subset \mathbb{R}$ we have*

$$\limsup_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \limsup_{n\to\infty} \frac{a_n}{b_n}.$$

Proof. Recall that $\limsup_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \sup_{m\geq n} \frac{a_m}{b_m}$. For any $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that for all $n \geq k$, $\sup_{m\geq n} \frac{a_m}{b_m} - L < \epsilon$. That is, $\sup_{m\geq k} \frac{a_m}{b_m} - L < \epsilon$. Fix ϵ, and then fix k. We have

$$\frac{a_m}{b_m} < L + \epsilon$$

for all $m \geq k$, so writing A_n, B_n be summation of the first n terms of $(a_n), (b_n)$ respectively,

$$\begin{aligned}
a_1 + \cdots + a_m &< a_1 + \cdots + a_k + (L+\epsilon)(b_{k+1} + \cdots + b_m) \\
&= a_1 + \cdots + a_k + (L+\epsilon)(b_1 + \cdots b_k + b_{k+1} + \cdots + b_m - (b_1 + \cdots b_k)) \\
A_m &< A_k + (L+\epsilon)(B_m - B_k) \\
\frac{A_m}{B_m} &< \frac{A_k}{B_m} + (L+\epsilon) - \frac{B_k}{B_m}(L+\epsilon) \\
\limsup_{m\to\infty} \frac{A_m}{B_m} &< \lim_{m\to\infty} \frac{A_k}{B_m} + (L+\epsilon) - \lim_{m\to\infty} \frac{B_k}{B_m}(L+\epsilon) \\
\limsup_{m\to\infty} \frac{A_m}{B_m} &< L+\epsilon
\end{aligned}$$

This is true for all $\epsilon > 0$ so the result follows. □

Remarks. Similarly, we can deduce a version for $\liminf$. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then its $\limsup, \liminf$ are the same.

Example 1. Calculate $\lim_{n\to\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{\ln n}$.

Proof. Read the above theorem in the form

$$\lim_{n\to\infty} \frac{A_n}{B_n} = \lim_{n\to\infty} \frac{A_n - A_{n-1}}{B_n - B_{n-1}},$$

then we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{\ln n} &= \lim_{n\to\infty} \frac{\frac{1}{n}}{\ln n - \ln(n-1)} \\
&= \lim_{n\to\infty} \frac{1}{\ln\left(\frac{n}{n-1}\right)^n} \\
&= \lim_{n\to\infty} \frac{1}{\ln\left(1 - \frac{1}{n}\right)^{-n}} \\
&= 1
\end{aligned}$$

□

Example 2. Calculate $\lim_{n\to\infty} \frac{n^n}{1 + 2^2 + 3^3 + \cdots + n^n}$.

Proof.

$$\begin{aligned}\lim_{n\to\infty} \frac{n^n}{1+2^2+3^3+\cdots+n^n} &= \lim_{n\to\infty} \frac{n^n-(n-1)^{n-1}}{n^n} \\ &= 1-\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{n-1}\frac{1}{n} \\ &= 1-\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}\frac{1}{n-1} \\ &= 1\end{aligned}$$

□

Example 3. Calculate $\lim\limits_{n\to\infty} nx_n$ where $x_1>0$ and $x_{n+1}=\ln(1+x_n)$.

Proof. Note that the sequence is positive and strictly decreasing, hence convergent. This gives $l := \lim_{n\to\infty} x_n = 0$ by using MATH1018.

This implies $(\frac{1}{x_n})$ is a strictly increasing sequence with limit ∞. By Stolz-Cesaro's theorem and repeated use of l'Hopital's theorem,

$$\begin{aligned}\lim_{n\to\infty} nx_n &= \lim_{n\to\infty}\frac{n}{\frac{1}{x_n}} \\ &= \lim_{n\to\infty}\frac{n+1-n}{\frac{1}{x_{n+1}}-\frac{1}{x_n}} \\ &= \lim_{n\to\infty}\frac{x_nx_{n+1}}{x_n-x_{n+1}} \\ &= \lim_{n\to\infty}\frac{x_n\ln(1+x_n)}{x_n-\ln(1+x_n)} \\ &= \lim_{n\to\infty}\frac{\ln(1+x_n)}{x_n}\frac{x_n^2}{x_n-\ln(1+x_n)} \\ &= \lim_{n\to\infty}\frac{x_n^2}{x_n-\ln(1+x_n)} \\ &= \lim_{n\to\infty}\frac{2x_n}{1-\frac{1}{1+x_n}} \\ &= \lim_{n\to\infty}\frac{2}{\frac{1}{(1+x_n)^2}} \\ &= 2\end{aligned}$$

□