THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2068 Honours Mathematical Analysis II

Tutorial 1

The following are discussed in the tutorial this week.

Let I be an open interval. Let $f : I \to \mathbb{R}$ be a function.

Definition.

- We say that f is differentiable at $c \in I$ if
$$f'(c) := \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$
exists. We call $f'(c)$ the derivative of f at c.
- We say that f is differentiable on I if $f'(x)$ exists for all $x \in I$.

Example 1. Suppose that f is differentiable at c and that $f(c) = 0$. Show that $g(x) := |f(x)|$ is differentiable at c if and only if $f'(c) = 0$.

Example 2. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by
$$g(x) := \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
Show that g is differentiable for all $x \in \mathbb{R}$. Also show that g' is not bounded on $[-1, 1]$.

Example 3. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, show that
$$f'(c) = \lim(n\{f(c + 1/n) - f(c)\}).$$
However, show by example that the existence of the limit of the sequence does not imply the existence of $f'(c)$.

Example 4. Consider the Thomae's function on $[0, 1]$:
$$h(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n}, \text{ for some positive integers } m, n \text{ with no common factors,} \\ 1, & \text{if } x = 0, \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$
Discuss the differentiability of h on $[0, 1]$.

Example 5. Let $f : (-a, a) \to \mathbb{R}$, with $a > 0$. Assume f is continuous at 0 and such that the limit
$$\lim_{x \to 0} \frac{f(x) - f(\lambda x)}{x} = l$$
exists, where $0 < \lambda < 1$. Show that $f'(0)$ exists. What happens to this conclusion when $\lambda > 1$.