MATH2068 Honour Mathematical Analysis II

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Claim 1. *Let $f : [a, b] \to [c, d]$ be integrable, $g : [c, d] \to \mathbb{R}$ be continuous, claim that $g \circ f$ is integrable on $[a, b]$.*

Proof. Fix $\epsilon > 0$, then there exists $\delta > 0$ such that $|g(p) - g(q)| < \frac{\epsilon}{2(b-a)}$ whenever $|p - q| < \delta$ by uniform continuity. Fix $\delta > 0$ then there exists a partition P on $[c, d]$ such that

$$\sum_i osc_i(f)||P_i|| < \frac{\epsilon\delta}{4M}$$

where $M = \max_{x\in[c,d]} g$, P_i means the i-th sub-interval and osc_i denotes the oscillation on P_i. Fix P.

$$\begin{aligned}\sum_i osc_i(g \circ f)||P_i|| &= \sum_{osc_i(f)\geq\delta} osc_i(g \circ f)||P_i|| + \sum_{osc_i(f)<\delta} osc_i(g \circ f)||P_i|| \\ &:= I + II.\end{aligned}$$

I:

$$\begin{aligned}\frac{\epsilon\delta}{4M} &> I \geq \delta \sum_{osc_i(f)\geq\delta} ||P_i|| \\ \therefore \frac{\epsilon}{4M} &\geq \sum_{osc_i(f)\geq\delta} ||P_i|| \\ \therefore I &< 2M\frac{\epsilon}{4M} \\ &\leq \frac{\epsilon}{2}\end{aligned}$$

II:

$$\begin{aligned}\sum_{osc_i(f)<\delta} osc_i(g \circ f)||P_i|| &< \frac{\epsilon}{2(b-a)} \sum_{osc_i(f)<\delta} ||P_i|| \\ &< \frac{\epsilon}{2(b-a)}(b-a) \\ &= \frac{\epsilon}{2}.\end{aligned}$$

Hence $U(g \circ f, P) - L(g \circ f, P) < \epsilon$ and integrability follows. □

Remarks. The proof feels like the proof for integrability of Thomae function, you separate a partition into two classes, one sum over-estimate by function and one sum over-estimate by total length.

Proposition 2. *Suppose f is bounded on $[a, b]$ and for any $c \in (a, b)$, the restriction $f|_{[c,b]}$ is integrable on $[c, b]$. Then f is integrable on $[a, b]$ and*

$$\lim_{c \to a^+} \int_c^b f = \int_a^b f.$$

Proof. Fix $\epsilon > 0$, let M such that $f \leq M$ on $[a, b]$, pick c such that $0 < c - a < \frac{\epsilon}{2M}$.

Define g, h to be equal to f on $[c, b]$ but on $[a, c]$ we have $g = -M, h = M$. g, h are integrable on $[a, b]$ and $\int_a^b (g - h) < \frac{\epsilon}{2M} 2M = \epsilon$. The results follows from squeeze theorem, which states that given a function f on $[a, b]$, if for all $\epsilon > 0$ there exists g, h such that $g \leq f \leq h$, g, h integrable on $[a, b]$ and $\int_a^b g - h < \epsilon$, then f is integrable on $[a, b]$.

For the limit, for all $\epsilon > 0$ we have $| \int_c^b f - \int_a^b f| < (c - a)M \to 0$ as $c \to a^+$. $\square$

Mid-term evaluation:

Q1: The quantity $\lim_{\delta\to 0}\sup_{0<x<\delta}\dfrac{f(x)}{x}$ needs to be calculated with demonstration. Besides, skipping proofs by saying "these different cases are similar" received mark deduction.

Q2: The function g is not said to be differentiable at a and b. All proofs assuming $g'(a), g'(b) \in \mathbb{R}$ received mark deduction to a large extent.

Q3: Q3 is not marked by me, but condition (5) is obviously insufficient because integrability a priori only talks to the function almost everywhere. However, if f is also continuous, then you can show $f \cong 0$.

The another half of tutorial content is about skip-proof analysis, or to be precise, how to write a proof for tests and exams.

Students often believe that they can skip-proof whenever a claim is trivial by consensus. For instance, students must have survived from MATH2058 with good grades in order to proceed to MATH2068, and then they often skip-proof whenever a claim looks too easy for an "ordinary A-range student in MATH2058".

However, we want your proof to be readable (as much as possible) by anyone with good math logic but not necessarily have taken the course. We discussed on the following example in class:

Claim 3. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function on $\mathbb{R}$ and M is a positive number such that $f(x) < M$ for all $x \in [a, b]$. Then there exists a $\delta > 0$ such that $f(x) < M$ for all $x \in [a + \delta, b + \delta]$.*

Remarks. Two common mistakes are taken by students in tests:

1. They would prefer to formulate the above claim like "if f is continuous with $f < M$ on a compact domain, then we can move the domain a little bit while preserving the estimate". (Note: this is convincing in a daily conversation, but don't do this in tests)

2. They gave no demonstration.

A detailed proof expects one to write down a possible δ explicitly, and demonstrate that for all $x \in [a + \delta, b + \delta]$ we indeed have the inequality $f(x) < M$.

We can write: by continuity of f at b, there exists $\delta > 0$ such that if $|y - x| < \delta$, then $|f(y) - f(x)| < \frac{M-\max_{x\in[a,b]} f}{2}$, where the RHS is positive by assumption. Then, for any $y \in [b, b + \frac{\delta}{2}]$, we have $f(y) - f(x) < \frac{M-\max_{x\in[a,b]} f}{2}$, which implies $f(y) < \frac{M-\max_{x\in[a,b]} f}{2} + f(x) \leq \frac{M-\max_{x\in[a,b]} f}{2} + \max_{x\in[a,b]} f = \frac{M+\max_{x\in[a,b]} f}{2} < M$.
Now we see that for any $y \in [a + \frac{\delta}{2}, b + \frac{\delta}{2}]$ the inequality $f(y) < M$ holds.

Remarks. There are something you can skip in tests:

- No need to write down the definitions, unless they are made up by you. For example, no need to tell me what is real number, function, continuity, limit, derivative, etc.
- No need to copy the proofs for classical theorems from Bartle. For example, no need to copy down the whole proof of Extreme Value Theorem.

Remarks. There are something you should double check in tests:

- Whether you checked the assumptions before you apply a theorem.
- Whether you state clearly on how you use a theorem; for example, some of you used Darboux theorem without telling me on which interval.
- (Just as a CUHK Math Alumni to say) Never assume hints are true statements; prove hints before you use them, if some hint are provided.

Remarks. The marking is more strict for take home exams, but we all know that face-to-face exams have less time and it is more unavoidable to skip-proof on easy facts. In that situation students may write down the most difficult, technical details first and see if they have time to make things clear.

Remarks. Later on when you take MATH3060, MATH4010, MATH4050, MATH4060 and etc, you also need to give lots of details. For example, in MATH3060 you are supposed to be familiar with how to make use of dense subspace in a function space (over $\mathbb{R}^n$), like using C_0^∞ dense in L^p to conclude that every L^p function can be approximated by a sequence of C_0^∞ function. And then you will learn about Lebesgue theory in MATH4050 and keep using step functions to do approximations. If you skip all the approximation arguments, you will be very likely to receive a bad grade.

I promised to write you an exposition to some grad-level mathematics for celebrating the π-day. If you are eligible for celebrating the White Valentine's Day, go ahead and don't let me know:)

Please note that I am not writing in the style of a solution for a test.

Theorem 4 (Riesz-Thorin interpolation theorem)**.** *Let* $1 \le p_0, p_1, q_0, q_1 \le \infty$ *and* $\theta \in (0,1)$. *Define* $1 \le p, q \le \infty$ *by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

and

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let T *be a linear map such that* $T : L^{p_0} \to L^{q_0}$ *is a linear operator with norm* $||T||_{p_0,q_0} = N_0$ *and* $T : L^{p_1} \to L^{q_1}$ *is a linear operator with norm* $||T||_{p_1,q_1} = N_1$.

Then, we claim that for any $f \in L^{p_0} \cap L^{p_1}$ *we have*

$$||Tf||_q \le N_0^{1-\theta} N_1^{\theta} ||f||_p.$$

Remarks. The result is trivial if $p = \infty$ or if $q = 1$ so I leave them as exercises.

We assume $p_0, p_1 < \infty$ such that the proof can contain a little bit fewer explanation.

Claim 5. *In the above setting, we have*

$$||f||_p \le ||f|||_{p_0}^{1-\theta} ||f||_{p_1}^{\theta}.$$

Proof.

$$\begin{aligned}
||f||_p^p &= \int |f|^p \\
&= \int |f|^{p\theta} |f|^{p(1-\theta)} \\
&\le \left(\int |f|^{p\theta \frac{p_1}{p\theta}} \right)^{\frac{p\theta}{p_1}} \left(\int |f|^{p(1-\theta)\frac{p_0}{p(1-\theta)}} \right)^{\frac{p(1-\theta)}{p_0}} \\
&= \left(\int |f|^{p_1} \right)^{\frac{p\theta}{p_1}} \left(\int |f|^{p_0} \right)^{\frac{p(1-\theta)}{p_0}} \\
&= ||f||_{p_1}^{p\theta} ||f||^{p(1-\theta)} p_0.
\end{aligned}$$

$\square$

Proof of Riesz-Thorin. For $1 \leq p_0, p_1 < \infty$, we know that continuous compactly supported functions are dense in $L^{p_0} \cap L^{p_1}$ with respect to the norm $||\cdot||_{L^{p_0} \cap L^{p_1}} = ||\cdot||_{p_0} + ||\cdot||_{p_1}$

By compact support, continuity can be extended to uniform continuity, so they can be approximated by compactly supported step-functions which have only finitely many values. To be more precise, fix ϵ, then ϵ defines a δ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$, compactness of domain implies finite open subcovering by balls with radius δ. Let ϵ be $\frac{1}{n}$ we deduce an approximating sequence.

Hence, by density (or we say: by approximation), it is enough to show that $||Tf||_q \leq N_0^{1-\theta} N_1^{\theta} ||f||_p$ for any compactly supported step function f.

We demonstrate what is actually happening behind the phrase ‘by density”. It means the following claim:

Claim 6. *If $||Tf||_q \leq N_0^{1-\theta} N_1^{\theta} ||f||_p$ for any compactly supported step function f, then the same inequality holds for all $f \in L^{p_0} \cap L^{p_1}$.*

Proof. Suppose the if part is true and $f \in L^{p_0} \cap L^{p_1}$, then we have already shown that there exists a sequence (f_n) of compactly supported step functions such that

$$f_n \to f$$

in $L^{p_0} \cap L^{p_1}$, which means

$$f_n \to f$$

in L^{p_0} and

$$f_n \to f$$

in L^{p_1}. Then we can write

$$\begin{aligned}
||Tf||_q &\leq ||T(f - f_n)||_q + ||Tf_n||_q \\
&\leq ||T(f - f_n)^{1-\theta}||_{q_0} ||T(f - f_n)||_{q_1}^{\theta} + N_0^{1-\theta} N_1^{\theta} ||f_n||_p \\
&\leq ||T(f - f_n)||_{q_0}^{1-\theta} ||T(f - f_n)||_{q_1}^{\theta} + N_0^{1-\theta} N_1^{\theta} ||f_n - f||_p + N^{1-\theta} N_1^{\theta} ||f||_p \\
&\leq ||f - f_n||_{p_0}^{1-\theta} N_0^{1-\theta} ||f - f_n||_{p_1}^{\theta} N_1^{\theta} + N_0^{1-\theta} N_1^{\theta} (||f||_p + ||f_n - f||_{p_0}^{1-\theta} ||f_n - f||_{p_1}^{\theta}) \\
&= N_0^{1-\theta} N_1^{\theta} (||f||_p + 2||f - f_n||_{p_0}^{1-\theta} ||f - f_n||_{p_1}^{\theta}) \\
&\xrightarrow{n\to\infty} N_0^{1-\theta} N_1^{\theta} ||f||_p.
\end{aligned}$$

$\square$

Now, our goal is to show $||Tf||_q \le N_0^{1-\theta}N_1^{\theta}||f||_p$ for any compactly supported step function f. We recall that by duality (MATH4010 I think) we have

$$||f||_q = \sup_{||g||_{q'}\le 1}\left|\int gf\right|$$

where $\frac{1}{q}+\frac{1}{q'}=1$.

Therefore, it reduces to show

$$\left|\int gTf\right| \le N_0^{1-\theta}N_1^{\theta}$$

for all compact supported step functions $||f||_p = ||g||_{q'} = 1$.

Write

$$f = \sum_{j=1}^{n} a_j\chi_{A_j}$$

where $a_1,\cdots,a_n \in \mathbb{C}$ and $A_1,\cdots,A_n$ are measurable disjoint subsets (at your level it assume χ_{A_i} is always integrable).

Similarly, we write

$$g = \sum_{l=1}^{m} b_l\chi_{B_l}.$$

Observe that $||f||_p^p = \sum_{j=1}^n |a_j|^p|A_j| = 1$ and $||g||_{q'}^{q'} = \sum_{l=1}^m |b_l|^{q'}|B_l|$.

For $z\in\mathbb{C}$ we define $p(z), q'(z)$ (not a derivative, just symbol q') by

$$\frac{1}{p(z)} = \frac{1-z}{p_0}+\frac{z}{p_1}$$

and

$$\frac{1}{q'(z)} = \frac{1-z}{q_0}+\frac{z}{q_1'}.$$

We have:

- $p(0)=p_0$, $q'(0)=q_0'$.
- $p(1)=p_1$, $q'(1)=q_1'$.
- $p(\theta)=p$, $q'(\theta)=q'$.

Define

$$f_z(x) = \begin{cases} |f(x)|^{\frac{p}{p(z)}}\frac{f(x)}{|f(x)|} & ; f(x)\neq 0 \\ 0 & ; f(x)=0 \end{cases}$$

$$g_z(x) = \begin{cases} |g(x)|^{\frac{q'}{q'(z)}} \frac{g(x)}{|g(x)|} & ; g(x) \neq 0 \\ 0 & ; g(x) = 0 \end{cases}$$

They are compact supported step functions. To be explicit, we expand the expression by the plugging in the definitions of f, g:

$$f_z(x) = \sum_{j=1}^{n} |a_j|^{\frac{p}{p(z)}} \frac{a_j}{|a_j|} \chi_{A_j}$$

$$g_z(x) = \sum_{l=1}^{m} |b_l|^{\frac{q'}{q'(z)}} \frac{b_l}{|b_l|} \chi_{B_l}$$

As finite sums of step functions, they are bounded.

Now, $Tf_z \in L^{q_0} \cap L^{q_1}$ is well-defined and can be integrable against the function $g_z \in L^{q_0'} \cap L^{q_1'}$.

We set $F(z) = \int g_z T f_z$, then

$$\begin{aligned} F(z) &= \sum_{j=1}^{n} \sum_{l=1}^{m} |a_j|^{\frac{p}{p(z)}} \frac{a_j}{|a_j|} |b_l|^{\frac{q'}{q'(z)}} \frac{b_l}{|b_l|} \int_{B_l} T\chi_{A_j} \\ &= \sum_{j=1}^{n} \sum_{l=1}^{m} |a_j|^{p(\frac{1-z}{p_0} + \frac{z}{p_1})} \frac{a_j}{|a_j|} |b_l|^{q(\frac{1-z}{q_0} + \frac{z}{q_1})} \frac{b_l}{|b_l|} \int_{B_l} T\chi_{B_l} \\ &= \sum_{j,l} C_{j,l} \gamma_j^z \gamma_l^z \end{aligned}$$

for some $C_{j,l}$ independent of z and $C_{j,l}, \gamma_j, \gamma_l \in \mathbb{C}$.

On $S = \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}$ we have $|\gamma_j^z| = \gamma^{\Re z} \leq \max\{1, \gamma\}$ and similarly for $|\gamma_l|$.

Wee see that F is bounded on S and continuous as a function of z, because it is just a finite sum of complex numbers raised to power z. Due to this simple expression, we see that F is also analytic in the interior S^0 of S.

We complete the proof by using Hadamard's three lines lemma:

Lemma 7 (Hadamard three lines lemma)**.** *Let $S = \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}$ and $F : S \to \mathbb{C}$ be bounded and continuous on S, and analytic in S^0. Let $M_\theta := \sup_{y \in \mathbb{R}} |F(\theta + iy)|$. Then we have*

$$M_\theta \leq M_0^{1-\theta} M_1^\theta \quad \forall \theta \in [0, 1].$$

Compute for M_0:

$$\begin{aligned}|F(iy)| &= \left|\int g_{iy} T f_{iy}\right| \\ &\leq ||Tf_{iy}||_{q_0} ||g_{iy}||_{q_0'} \\ &\leq N_0 ||f_{iy}||_{p_0} ||g_{iy}||_{q_0'}\end{aligned}$$

look into the two norm-terms:

$$\begin{aligned}||f_{iy}||_{p_0}^{p_0} &= \sum \left| |a_j|^{\frac{p}{p_{iy}}} \frac{a_j}{|a_j|} \right|^{p_0} |A_j| \\ &= \sum \left| |a_j|^{\Re \frac{p}{p_{iy}}} \frac{a_j}{|a_j|} \right|^{p_0} |A_j| \\ &= \sum |a_j|^p |A_j| \\ &= ||f||_p^p\end{aligned}$$

because for $\frac{1}{p(iy)} = \frac{1-iy}{p_0} + \frac{iy}{p_1}$, the real part is $\frac{1}{p_0}$ and for complex numbers w, real number r in general, $|r^w| = |r^{\Re w + i\Im w}| = |r^{\Re w} e^{ir \ln \Im w}| = |r^{\Re w}|$ (if $\Im w = 0$ then we arrive at the same conclusion).

Similarly,

$$\begin{aligned}||g_{iy}||_{q_0'}^{q_0'} &= \sum \left| |b_l|^{\frac{q'}{q'(iy)}} \frac{b_l}{|b_l|} \right|^{q_0'} |B_l| \\ &= \sum \left| |b_l|^{\Re \frac{q'}{q'(iy)}} \frac{b_l}{|b_l|} \right|^{q_0} |B_l| \\ &= \sum |b_l|^{q'} |B_l| \\ &= ||g||_{q'}^{q'}.\end{aligned}$$

Hence, $M_0 \leq N_0 ||f_{iy}||^{\frac{p}{p_0}} ||g_{iy}||_{q_0'}^{\frac{q'}{q_0'}}$.

Compute for M_1:

Again, we note that for $\frac{1}{p(1+iy)} = \frac{1-1-iy}{p_0} + \frac{1+i}{p_1}$, the real part is $\Re \frac{1}{p(1+iy)} = \frac{1}{p_1}$. We can deduce that

$$||f_{1+iy}||_{p_1}^{p_1} = ||f||_p^p$$

and

$$||g_{1+iy}||_{q_1'}^{q_1'} = ||g||_{q'}^{q'}.$$

So, $M_1 \leq N_1 ||f||_p^{\frac{p}{p_1}} ||g||^{\frac{q'}{q'_1}}$

By the three lines lemma, we have $M_\theta \leq M_0^{1-\theta} M_1^\theta$, when $z = \theta$ we have $F(\theta) = \int g_\theta T f_\theta = \int gTf$, recall $||g||_{q'} = 1$, so combine everything above we obtain

$$||Tf||_q \leq N_0^{1-\theta} N_1^\theta ||f||_p^{\frac{p}{p+1}(1-\theta)+\frac{p}{p_1}\theta} = N_0^{1-\theta} N_1^\theta |f||_p.$$

$\square$

An application of Riesz-Thorin's theorem: if you can estimate $||f||_\infty$ by $||u||_1$ and $||u||_\infty$ for some u, then you can estimate $||f||_\infty$ by $||u||_p$ for any $1 < p < \infty$ with a good knowledge on the coefficient.

When the domain is compact (finite measure) it might not sound useful to estimate by $||u||_p$ when we can already estimate by $||u||_1$. However, the p can be useful when we want to prove some linear operator is of weak type (p, q), $\frac{1}{p} + \frac{1}{q} = 1$, see arXiv:0707.2424 section 9 (you can skip section 1-8).