MATH2068 Honour Mathematical Analysis II

Mid-term solution

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Typos are unavoidable. Please let me know if you spot any.

- Q1:
  - Observe that D^+ is taking limit from the right and D^- is taking limit from the left. We may assume $D^+f(c) \geq D^-f(c)$ because, if this is not the case, we consider the function $f(-x)$ instead of $f(x)$. This reflects the graph with respect to the y-axis, which means left and right interchanges.

    We claim that for any $\epsilon > 0$, there exists $\gamma > 0$ such that if $0 < \delta < \gamma$ then

    $$\sup_{0<|x-c|<\delta} \frac{f(x)-f(c)}{x-c} \leq D^+f(c) + \epsilon.$$

    Once we have the claim, let $\delta \to 0$ we obtain

    $$\lim_{\delta\to 0} \sup_{0<|x-c|<\delta} \frac{f(x)-f(c)}{x-c} \leq D^+f(c) = \max\{D^+f(c), D^-f(c)\}. \qquad (1)$$

    Proof of claim: By definition, $D^+f(c) := \lim_{\delta\to 0^+} \sup_{c<x<c+\delta} \frac{f(x)-f(c)}{x-c}$. For any $\epsilon > 0$, there exists $\gamma_1 > 0$ such that if $0 < \delta < \gamma_1$ then

    $$\sup_{c<x<c+\delta} \frac{f(x)-f(c)}{x-c} - D^+f(c) \leq |D^+f(c) - \sup_{c<x<c+\delta} \frac{f(x)-f(c)}{x-c}| < \delta.$$

    $$\sup_{c<x<c+\delta} \frac{f(x)-f(c)}{x-c} < D^+f(c) + \delta. \qquad (2)$$

    By definition, $D^-f(c) := \lim_{\delta\to 0^+} \sup_{c-\delta<x<c} \frac{f(x)-f(c)}{x-c}$. For any $\epsilon > 0$, there exists $\gamma_2 > 0$ such that if $0 < \delta < \gamma_2$ then

    $$\sup_{c-\delta<x<c} \frac{f(x)-f(c)}{x-c} - D^-f(c) \leq |D^-f(c) - \sup_{c-\delta<x<c} \frac{f(x)-f(c)}{x-c}| < \delta.$$

    $$\sup_{c-\delta<x<c} \frac{f(x)-f(c)}{x-c} < D^-f(c) + \delta \leq D^+f(c) + \delta. \qquad (3)$$

    Choose $\gamma = \min\{\gamma_1, \gamma_2\}$, if $0 < \delta < \gamma$ then (2), (3) holds at the same time. That is, for any such δ, $D^+f(c) + \delta$ is an upper bound for $\frac{f(x)-f(c)}{x-c}$ on $(c-\delta, c)$, and is an upper bound for the same ratio on $(c, c+\delta)$.

    Note that $(c-\delta, c) \cup (c, c+\delta) = \{x : 0 < |x-c| < \delta\}$, so $D^+f(c) + \delta$ is an upper bound for $\frac{f(x)-f(c)}{x-c}$ on the LHS set = RHS set, i.e. $\{x : 0 < |x-c| < \delta\}$. We obtained

    $$\sup_{0<|x-c|<\delta} \frac{f(x)-f(c)}{x-c} \leq D^+f(c) + \delta,$$

    our claim follows.

Next, we claim that the opposite direction of (1) is true, i.e.

$$\lim_{\delta\to 0}\sup_{0<|x-c|<\delta}\frac{f(x)-f(c)}{x-c} \geq D^+f(c) = \max\{D^+f(c), D^-f(c)\}. \tag{4}$$

Proof of claim: if M is an upper bound for $\frac{f(x)-f(c)}{x-c}$ on $0<|x-c|<\delta$, then since every point satisfying $0<|x-c|<\delta$ must satisfy $c<x<c+\delta$, M is an upper bound for $\frac{f(x)-f(c)}{x-c}$ on $c<x<c+\delta$. Put $M=\sup_{c<x<c+\delta}\frac{f(x)-f(c)}{x-c}$, we have for all $c<x<c+\delta$:

$$\sup_{0<|x-c|<\delta}\frac{f(x)-f(c)}{x-c} \geq \frac{f(x)-f(c)}{x-c}$$

and hence

$$\sup_{0<|x-c|<\delta}\frac{f(x)-f(c)}{x-c} \geq \sup_{c<x<c+\delta}\frac{f(x)-f(c)}{x-c}$$

Take $\delta\to 0^+$ the RHS is called $D^+f(c)$, so we have

$$D^+f(c) = \lim_{\delta\to 0^+}\sup_{c<x<c+\delta}\frac{f(x)-f(c)}{x-c} \leq \lim_{\delta\to 0^+}\sup_{0<|x-c|<\delta}\frac{f(x)-f(c)}{x-c}.$$

Combine two claim the inequality signs can be replaced by $=$. Q1a follows.

- For part (b), the limits are $D^+f(0)=b$, $D_+f(0)=a$, $D^-f(0)=d$, $D_-f(0)=c$. Those who lost marks in part (b) just simply write down the limits, so I think you all know the proof.

  Make use of sequences such as $\{\frac{1}{2n\pi+\frac{\pi}{2}}\}$ and that kind of stuff.

- Q2: Suppose g is continuous on $[a,b]$ and g is differentiable on (a,b). Due to this assumption, we can use mean value theorem: for any $p<q\in(a,b)$ there exists $c\in(p,q)$ such that $g(p)-g(q)=g'(c)(p-q)$.

  Let D be the set of points where $g'(x)$ is non-zero.

  Suppose D is non-empty and countable. By non-empty, there exists $d\in D$ such that $s:=g'(d)\neq 0$. By countability and (a,b) is not countable as a continuum of points, D is a proper subset of (a,b). Hence, there exists $r\in(a,b)\setminus D$. Without loss of generality, we assume $d<r$.

  $[d,r]$ is a proper subset of $[a,b]$ so the continuity of g on $[a,b]$ inherits to $[d,r]$. Also, (d,r) is a proper subset of (a,b) so the differentiability of g on (a,b) inherits to (d,r). Hence, Darboux theorem can be applied on g on $[d,r]$, i.e. for any $z\in(0,s)$ there exists $w\in(d,r)$ such that $g'(w)=z$.

  w can be not unique, it's fine. Since we can assign a w for each given z, we consider the mapping $z\mapsto w(z)$. For any $z_1,z_2\in(d,r)$, if $w(z_1)=w(z_2)$ then $z_1=g'(w(z_1))=g'(w(z_2))=z_2$, so the map is injective. This implies D contains a continuum of image set, contradicts countability.

  Hence, D cannot be non-empty.

  Now we know $D=\emptyset$ (which is countable) and hence g has zero derivative everywhere on (a,b). In particular, the c suggested at the beginning must give $g'(c)=0$. For any $p<q\in(a,b)$, mean value theorem says there exists $c\in(p,q)$ such that $g(p)-g(q)=g'(c)(p-q)=0$. hence $g(p)=g(q)$ and g is constant.

- Q3: Counter-example: let $f(x) = \delta_a$, i.e. $f(x) = 1$ when $x = a$ and $f(x) = 0$ if otherwise. The function $x^n \delta_a$ is equal to $a^n \delta_a$ and you can show that it integrates to 0.

  We claim that if f is continuous and integrable on $[a, b]$, then under the condition (5) in the test paper, $f \cong 0$ holds.

  Assume Weierstrass Approximation theorem, i.e. for such f, for any $\epsilon > 0$ there exists a polynomial such that $|f(x) - p(x)| < \epsilon$ for all $x \in [a, b]$.

  The main idea is, $\int f^2 = \int fp = 0$ by condition (5). If f is not the zero function, say $f(c) > 0$ for some $c \in (a, b)$, then there exists an open interval I containing c and contained by (a, b) such that $f \geq k > 0$ for some constant k (Note: $k = \inf_{x \in I} f(x)$). This gives $\int_a^b f^2 \geq |I|k^2 > 0$, contradiction arises.

  To be precise, write $p = \sum_n a_n x^n$, then

  $$\int f^2 = \int fp = \sum_n a_n \int f x^n = 0$$

  by condition (5). Write $I = (c - \delta, c + \delta)$ where δ is chosen by: since $\frac{1}{2} f(c) > 0$, continuity of f at c says there exists δ such that if $0 < |x - c| < \delta$ then $f(x) - f(c) \leq |f(x) - f(c)| < \frac{1}{2} f(c)$. Now let $I = (c - \delta, c + \delta)$, we have $k = \frac{1}{2} f(c)$ as in the main idea, Finally, $\int f^2 \geq \int_I f^2 \geq |I|k^2 = 2\delta k^2 > 0$.