MATH2068 Honour Mathematical Analysis II

Bounded Convergence Theorem

Clive Chan

Theorem 1. *Suppose $\{f_n\}$ is a sequence of $[a,b] \to \mathbb{R}$ functions such that:*

- *There exists B independent of n such that $|f_n| \leq B$ for all n; and*
- *f_n and f are all Riemann-integrable.*

Then

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

Proof. Let $g_n = |f - f_n|$, then

$$|\int_a^b f - \int_a^b f_n| \leq \int_a^b |f - f_n| = \int_a^b g_n,$$

$g_n \geq 0$, $g_n \leq 2B$ and g_n is Riemann-integrable on $[a,b]$.

Assume $\{g_n\}$ is decreasing, i.e. $\{f_n\}$ is an increasing sequence.

Fix $\epsilon > 0$, Let h_n be a continuous function such that

$$\int_a^b g_n < \int_a^b h_n + \frac{\epsilon}{2^n}$$

and $0 \leq h_n \leq g_n$. The construction of h_n is left for the students.

By monotonicity,

$$0 \leq g_n - h_n \leq \sum_{i=1}^{n} g_i - h_i$$

which implies

$$0 \leq \int_a^b g_n - h_n \mathrm{d}x \leq \sum_i \int_a^b g_i - h_i \leq \sum_i \frac{\epsilon}{2^i} = \epsilon(1 - \frac{1}{2^n}).$$

$\{h_n\}$ is continuous, pointwise limit tends to 0 which is continuous on $[a,b]$ and the sequence is monotone decreasing to 0. By Dini's theorem $\{h_n\}$ converges to 0 uniformly on $[a,b]$. Hence

$$\lim_{n\to 0} \int_a^b h_n = 0.$$

This forces

$$\int_a^b g_n \mathrm{d}x = 0$$

and hence

$$\int_a^b f_n \to \int_a^b f.$$

Consider the general case where $\{g_n\}$ is not monotone: define $p_n(x) = \sup_{k\geq 0} g_{n+k}(x)$, then $\{p_n\}$ is monotone decreasing. Repeat the above argument (left for the students). □

Remarks. My first insight did not lead to this proof. I defined

$$S_n = \{x \in [a, b] : f \text{ or } f_n \text{ discontinuous at } x \ \exists n\}$$

and cover $\cup_n S_n$ with open intervals with total length less than ϵ. Since f and f_n are Riemann-integrable, S_n has measure 0 due to Lebesgue integrability criterion. Denote the open cover by $\cup_n I_n$, $[a, b] - I_n$ is compact and hence f, f_n are (uniformly) continuous on $[a, b] - I_n$.

We may define $g_n = |f - f_n|$ and $p_n = \sup_{k\geq 0} g_{n+k}$ such that $\{p_n\}$ is monotone decreasing to 0. Construct continuous approximations $\{h_n \leq p_n\}$ as above, we can invoke Dini's theorem to deduce

$$\int_a^b h_n \to 0$$

and hence the result follows.

Special thanks to student Mr. Kai-Kwan Lau for interesting discussions on this method.

An alternate proof. The uniform upper bound B in the above bounded convergence theorem may be replaced by an integrable function (instead of being a constant). In the Lebesgue setting, this is known as the Lebesgue Dominated Convergence Theorem.

Assume measure theory, we shall prove like this:

Fix $\epsilon > 0$. Let

$$E_{n,k} = \{x \in [a, b] : |f_n(x) - f(x)| < \frac{1}{k}\},$$

then pointwise convergence means:

For all $k \in \mathbb{N}$, pick $x \in [a, b]$, then there exists some $N \in \mathbb{N}$ such that if $n > \mathbb{N}$ then

$$|f_n(x) - f(x)| < \frac{1}{k}.$$

In other words, for any $k \in \mathbb{N}$,

$$[a, b] = \cup_{N\in\mathbb{N}} \cap_{n>N} E_{n,k} = \lim_{N\to\infty} \cap_{n>N} E_{n,k}.$$

where the latter equations comes from nested sequence. Hence, there exists $N(k)$ such that

$$\mu\{[a, b] - \cap_{n>N(k)} E_{n,k}\} < \frac{\epsilon}{2^k}.$$

If

$$x \in \cap_k \cap_{n>N(k)} E_{n,k},$$

then for all k (there exists $N(k)$) for all $n > N(k)$ we have

$$|f_n(x) - f(x)| < \frac{1}{k}.$$

In other words, on $\cap_k \cap_{n>N(k)} E_{n,k}$ we have $f_n \to f$ uniformly.

The measure of $[a, b] - \cap_k \cap_{n>N(k)} E_{n,k} = \cup_k([a, b] - \cap_{n>N(k)} E_{n,k}) \leq \sum_k \frac{\epsilon}{2^k} = \epsilon$ is small. Hence, for [exercise: fill in this square bracket to complete the proof] we have

$$\left|\int_a^b f_n - f\right| \leq \int_a^b |f_n - f| \leq (b - a)\epsilon + 2B\epsilon$$

and the result follows. □

Corollary 2 (Fatou's lemma for the Riemann integral)**.** *Suppose a non-negative sequence of Riemann-integrable functions* $\{f_n\}$ *converge pointwisely to some* $0 \leq f \in R[a,b]$*, then*

$$0 \leq \int_a^b f(x)\mathrm{d}x \leq \liminf_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x$$

Proof. $f(x) - f_n(x) \leq f(x)$ for all $x \in [a,b]$, so $(f(x) - f_n(x))^+ \leq f(x)$. By the dominated convergence theorem.

$$\lim_{n\to} \int_a^b (f(x) - f_n(x))^+ \mathrm{d}x = 0.$$

Write $f = (f - f_n) + f_n \leq (f - f_n)^+ + f_n$ for all n, so

$$\int_a^b f(x)\mathrm{d}x \leq \liminf_{n\to\infty} \left(\int_a^b (f(x) - f_n(x))^+ \mathrm{d}x + \int_a^b f_n(x)\mathrm{d}x \right) = \liminf_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x$$

where we see that the first limit vanishes. □

Remarks. Refer to:

W. A. J. Luxemburg (1971) Arzelá's Dominated Convergence Theorem for the Riemann Integral, The Amerizan Mathematical Monthly, 78:9, 970-979, DOI:10.1080/00029890.1971.11992915