MATH 2068: Honours Mathematical Analysis II: Home Test 1
5:00 pm, 01 Mar 2024

Important Notice:

♣ The answer paper **must be submitted before 02 Mar 2024 at 5:00 pm.**
♠ The answer paper MUST BE sent to the CU Blackboard.
✠ **The answer paper must include your name and student ID.**
§ **You are not allowed to resubmit the answer paper again after doing the first submission**
Answer ALL Questions

1. **(15 points)**

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. For each point $c \in \mathbb{R}$, define the following limits provided they exist.

$$D^+f(c) := \overline{\lim_{x \to c+}} \frac{f(x) - f(c)}{x - c} := \lim_{\delta \to 0+} \sup_{c < x < c+\delta} \frac{f(x) - f(c)}{x - c}; \tag{1}$$

$$D_+f(c) := \underline{\lim_{x \to c+}} \frac{f(x) - f(c)}{x - c} := \lim_{\delta \to 0+} \inf_{c < x < c+\delta} \frac{f(x) - f(c)}{x - c};. \tag{2}$$

Similarly, one can naturally define the following notation.

$$D^-f(c) := \overline{\lim_{x \to c-}} \frac{f(x) - f(c)}{x - c}; \quad \textit{and} \quad D_-f(c) := \underline{\lim_{x \to c-}} \frac{f(x) - f(c)}{x - c}. \tag{3}$$

(i) Show that

$$\max(D^+f(c), D^-f(c)) = \overline{\lim_{x \to c}} \frac{f(x) - f(c)}{x - c} := \lim_{\delta \to 0+} \sup_{0 < |x-c| < \delta} \frac{f(x) - f(c)}{x - c}.$$

(ii) Let $a < b$ and $c < d$. Put

$$f(x) := \begin{cases} ax(\sin \frac{1}{x})^2 + bx(\cos \frac{1}{x})^2 & \text{if } x > 0; \\ 0 & \text{if } x = 0; \\ cx(\sin \frac{1}{x})^2 + dx(\cos \frac{1}{x})^2 & \text{if } x < 0. \end{cases} \tag{4}$$

Find $D^+f(0); D_+f(0); D_-f(0)$ and $D^-f(0)$.

2. **(10 points)**
Assume that a function g is continuous on $[a, b]$ and is differentiable on (a, b). If the set $D := \{x \in (a, b) : g'(x) \neq 0\}$ is countable, does it imply that g is a constant function?

3. **(15 points)**
Let f be a Riemann integrable function defined on $[a, b]$. Consider the following condition:

$$\int_a^b x^n f(x)dx = 0 \quad \text{for all positive integers } n = 0, 1, 2.... \tag{5}$$

Discuss the following situation.
Is $f \equiv 0$ on $[a, b]$ under the Condition(5)?
If it is not the case, do you think that which class of functions on $[a, b]$ under the Condition(5) so that $f \equiv 0$ holds?

$* * *$ **END OF PAPER** $* * *$