MATH 2068 Mathematical Analysis II
2023-24 Term 2
Suggested Solution to Homework 8

8.2-1 Show that the sequence $(x^n/(1+x^n))$ does not converge uniformly on $[0,2]$ by showing that the limit function is not continuous on $[0,2]$.

Solution. It is easy to see that

$$\lim\left(x^n/(1+x^n)\right) = f(x) := \begin{cases} 0 & \text{if } 0 \le x < 1; \\ 1/2 & \text{if } x = 1; \\ 1 & \text{if } 1 < x \le 2. \end{cases}$$

If the sequence $(x^n/(1+x^n))$ converges uniformly on $[0,2]$, then the limit function f is also continuous on $[0,2]$, by Theorem 8.2.2 of the textbook. However, f is clearly discontinuous at 1. Therefore the sequence $(x^n/(1+x^n))$ does not converge uniformly on $[0,2]$. □

8.2-8 Let $f_n(x) := nx/(1+nx^2)$ for $x \in A := [0,\infty)$. Show that each f_n is bounded on A, but the pointwise limit f of the sequence is not bounded on A. Does (f_n) converge uniformly to f on A?

Solution. By the inequality $2xy \le x^2 + y^2$ for $x, y \in \mathbb{R}$, we have, for any $n \in \mathbb{N}$,

$$0 \le f_n(x) = \frac{nx}{1+nx^2} \le \frac{1}{2} \cdot \frac{1+n^2x^2}{1+nx^2} \le \frac{n+n^2x^2}{1+nx^2} = n \quad \text{for any } x \in A.$$

Hence each f_n is bounded on A.

On the other hand, the pointwise limit of the sequence is

$$\lim f_n(x) = \lim\left(\frac{nx}{1+nx^2}\right) = \lim\left(\frac{x}{1/n+x^2}\right) = f(x) := \begin{cases} 0 & \text{if } x = 0; \\ 1/x & \text{if } x > 0. \end{cases}$$

Clearly f is unbounded on A since $\lim\limits_{x\to 0^+} 1/x = +\infty$.

Finally (f_n) does not converge uniformly to f on A. Otherwise, by the continuity of each f_n and Theorem 8.2.2 in the textbook, f must be continuous on A, which is impossible. □

8.2-12 Show that $\lim \int_1^2 e^{-nx^2}\,dx = 0$.

Solution. Note that for all $n \in \mathbb{N}$ and $x \in [1,2]$,

$$0 \le e^{-nx^2} \le \frac{1}{nx^2} \le \frac{1}{n}.$$

Thus the sequence of continuous functions $\left(e^{-nx^2}\right)$ converges uniformly to the zero function on $[1,2]$. By Theorem 8.2.4 in the textbook,

$$\lim \int_1^2 e^{-nx^2}\,dx = \int_1^2 0\,dx = 0.$$

□