MATH 2068 Mathematical Analysis II
2023-24 Term 2
Suggested Solution to Homework 7

8.1-4 Evaluate $\lim\left(x^n/(1+x^n)\right)$ for $x \in \mathbb{R}$, $x \geq 0$.

Solution. If $0 \leq x < 1$, then $\lim\left(x^n/(1+x^n)\right) = 0/(1+0) = 0$.

If $x = 1$, then $\lim\left(x^n/(1+x^n)\right) = 1/(1+1) = 1/2$.

If $x > 1$, then $\lim\left(x^n/(1+x^n)\right) = \lim\left(1/(x^{-n}+1)\right) = 1/(0+1) = 1$. □

8.1-14 Show that if $0 < b < 1$, then the convergence of the sequence in Exercise 4 is uniform on the interval $[0, b]$, but is not uniform on the interval $[0, 1]$.

Solution. Let (f_n) be the sequence of functions considered in Exercise 4, and let f be its limit. Since $0 \leq f_n(x) = \frac{x^n}{1+x^n} \leq \frac{b^n}{1+0} = b^n$ for any $x \in [0, b]$, we have

$$\|f_n - 0\|_{[0,b]} \leq b^n \quad \text{for all } n \in \mathbb{N}.$$

As $0 < b < 1$, we have $\lim(b^n) = 0$ and so $\lim \|f_n - 0\|_{[0,b]} = 0$. Therefore (f_n) converges uniformly to $f \equiv 0$ on $[0, b]$.

On the other hand, for all $n \in \mathbb{N}$,

$$\|f_n - f\|_{[0,1]} \geq |f_n(2^{-1/n}) - f(2^{-1/n})| = \left|\frac{1/2}{1+1/2} - 0\right| = \frac{1}{3}.$$

So $\|f_n - f\|_{[0,1]} \not\to 0$ as $n \to \infty$. Therefore (f_n) does not converge uniformly to f on $[0, 1]$. □

8.1-22 Show that if $f_n(x) := x + 1/n$ and $f(x) := x$ for $x \in \mathbb{R}$, then (f_n) converges uniformly on $\mathbb{R}$ to f, but the sequence (f_n^2) does not converge uniformly on $\mathbb{R}$. (Thus the product of uniformly convergent sequences of functions may not converge uniformly.)

Solution. Since $\|f_n - f\|_{\mathbb{R}} = 1/n \to 0$ as $n \to \infty$, (f_n) converges uniformly on $\mathbb{R}$ to f.

On the other hand, for all $n \in \mathbb{N}$,

$$f_n^2(x) - f^2(x) = \left(x + \frac{1}{n}\right)^2 - x^2 = \frac{2x}{n} + \frac{1}{n^2} \quad \text{for any } x \in \mathbb{R},$$

so that

$$\|f_n^2 - f^2\|_{\mathbb{R}} \geq |f_n^2(n) - f^2(n)| = 2 + \frac{1}{n^2} \geq 2.$$

Therefore (f_n^2) does not converge uniformly to f^2 on $\mathbb{R}$. And so (f_n^2) does not converge uniformly on $\mathbb{R}$ because f^2 is the pointwise limit of (f_n^2). □