MATH 2068 Mathematical Analysis II
2023-24 Term 2
Suggested Solution to Homework 6

7.3-15 If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $c > 0$, define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := \int_{x-c}^{x+c} f(t)\,dt$. Show that g is differentiable on $\mathbb{R}$ and find $g'(x)$.

Solution. Since f is continuous on $\mathbb{R}$, it is Riemann integrable on any closed bounded interval by Proposition 2.13. By Proposition 2.18, for any $x \in \mathbb{R}$,

$$g(x) = \int_{x-c}^{x+c} f(t)\,dt = \int_{c}^{x+c} f(t)\,dt - \int_{c}^{x-c} f(t)\,dt.$$

Since f is continuous on any closed bounded interval $[a,b]$, Fundamental Theorem of Calculus (Theorem 2.25(ii)) implies that $F(x) := \int_c^x f(t)\,dt$ is differentiable on (a,b) with $F' = f$ on (a,b). As this is true for any $[a,b] \subseteq \mathbb{R}$, F is differentiable on $\mathbb{R}$. It then follows from Chain Rule (Proposition 1.6) that $g(x) = F(x+c) - F(x-c)$ is also differentiable on $\mathbb{R}$ and that

$$g'(x) = F'(x+c)\cdot(x+c)' - F'(x-c)\cdot(x-c)' = f(x+c) - f(x-c) \quad \text{for } x \in \mathbb{R}.$$

□

7.3-17 Let $J := [\alpha, \beta]$, let $\varphi : J \to \mathbb{R}$ have a continuous derivative on J, and let $f : I \to \mathbb{R}$ be continuous on an interval I containing $\varphi(J)$.

Use the following argument to prove the Substitution Theorem 7.3.8.

Define $F(u) := \int_{\varphi(\alpha)}^{u} f(x)\,dx$ for $u \in I$, and $H(t) := F(\varphi(t))$ for $t \in J$. Show that $H'(t) = f(\varphi(t))\varphi'(t)$ for $t \in J$ and that

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = F(\varphi(\beta)) = H(\beta) = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)\,dt.$$

Solution. Write $I = [a,b]$. Since f is continuous on $[a,b]$, Fundamental Theorem of Calculus (Theorem 2.25(ii)) implies that $F(u)$ is differentiable on (a,b) and $F' = f$ on (a,b). By Chain Rule (Proposition 1.6), $H = F \circ \varphi$ is differentiable on J and

$$H'(t) = F'(\varphi(t))\varphi'(t) \quad \text{for } t \in J.$$

Hence, by Fundamental Theorem of Calculus (Theorem 2.25(i)) again,

$$\int_{\alpha}^{\beta} F'(\varphi(t))\varphi'(t)\,dt = \int_{\alpha}^{\beta} H'(t)\,dt = H(\beta) - H(\alpha).$$

Since $H(\alpha) = \int_{\varphi(\alpha)}^{\varphi(\alpha)} f(x)\,dx = 0$, we have

$$\int_{\alpha}^{\beta} F'(\varphi(t))\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx.$$

□