MATH 2068 Mathematical Analysis II
2023-24 Term 2
Suggested Solution to Homework 5

7.2-12 Show that $g(x) := \sin(1/x)$ for $x \in (0,1]$ and $g(0) := 0$ belongs to $\mathcal{R}[0,1]$.

Solution. Clearly $|g(x)| \leq 1$ for all $x \in [0,1]$.

Let $\varepsilon > 0$. Choose $c \in (0,1)$ such that $c < \varepsilon/4$. On $[c,1]$, $g(x) = \sin(1/x)$ is continuous, and hence $g \in \mathcal{R}[c,1]$ by Proposition 2.13. By Theorem 2.10, there is a partition $P : c = x_0 < x_1 < \cdots < x_n = 1$ on $[c,1]$ such that

$$0 \leq U(g,P) - L(g,P) = \sum_{i=1}^{n} \omega_i(g,P)\Delta x_i < \varepsilon/2,$$

where $\omega_i(g,P) := \sup\{|g(x) - g(x')| : x, x' \in [x_{i-1}, x_i]\}$. Now $P' : 0 =: x_{-1} < x_0 = c < x_1 < \cdots < x_n = 1$ is a partition on $[0,1]$ that satisfies

$$\begin{aligned} 0 \leq U(g,P') - L(g,P') &= \sum_{i=0}^{n} \omega_i(g,P')\Delta x_i \\ &= \sup\{|g(x) - g(x')| : x, x' \in [0,c]\}(c-0) + \sum_{i=1}^{n} \omega_i(g,P)\Delta x_i \\ &< 2(\varepsilon/4) + \varepsilon/2 = \varepsilon. \end{aligned}$$

By Theorem 2.10 again, $g \in \mathcal{R}[0,1]$. □

7.3-22 Let $h : [0,1] \to \mathbb{R}$ be Thomae's function and let sgn be the signum function. Show that the composite function $\operatorname{sgn} \circ h$ is not Riemann integrable on $[0,1]$.

Solution. Note that the composite function $\operatorname{sgn} \circ h$ is Dirichlet's function, which is given by

$$f(x) := \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0,1] \backslash \mathbb{Q}. \end{cases}$$

We will show that $f \notin \mathcal{R}[0,1]$. Let $P : 0 = x_0 < x_1 < \cdots < x_n = 1$ be an arbitrary partition on $[0,1]$. Since both rational numbers and irrational numbers are dense, for each $i = 1, \ldots, n$, we can find $r_i \in \mathbb{Q}$ and $q_i \in \mathbb{R} \backslash \mathbb{Q}$ such that $r_i, q_i \in (x_{i-1}, x_i)$. Then

$$\omega_i(f,P) = \sup\{|f(x) - f(x')| : x, x' \in [x_{i-1}, x_i]\} \geq |f(r_i) - f(q_i)| = 1 \quad \text{for } i = 1, \ldots, n,$$

and so

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} \omega_i(f,P)\Delta x_i \geq \sum_{i=1}^{n} \Delta x_i = 1.$$

By Theorem 2.10, $f \notin \mathcal{R}[0,1]$.

□