MATH 2068 Mathematical Analysis I
2023-24 Term 2
Suggested Solution to Homework 4

7.2-8 Suppose that f is continuous on $[a, b]$, that $f(x) \geq 0$ for all $x \in [a, b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution. Suppose $f(x_0) > 0$ for some $x_0 \in [a, b]$. Since f is continuous on $[a, b]$, we can further assume that $x_0 \in (a, b)$. Then there is $\delta > 0$ such that $a < x_0 - \delta < x_0 + \delta < b$ and that

$$f(x) > f(x_0)/2 =: m \quad \text{for any } x \in (x_0 - \delta, x_0 + \delta).$$

Now, by Proposition 2.14 and 2.18, we have

$$\begin{aligned}\int_a^b f &= \int_a^{x_0-\delta} f + \int_{x_0-\delta}^{x_0+\delta} f + \int_{x_0+\delta}^b f \\ &\geq 0 + \int_{x_0-\delta}^{x_0+\delta} m + 0 \\ &= 2m\delta > 0,\end{aligned}$$

which is a contradiction. □

7.2-10 If f and g are continuous on $[a, b]$ and if $\int_a^b f = \int_a^b g$, prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

Solution. Suppose $f(x) \neq g(x)$ for any $x \in [a, b]$. Then the Intermediate Value Theorem implies that either $f - g > 0$ or $g - f > 0$ on $[a, b]$. Together with $\displaystyle\int_a^b (f - g) = \int_a^b f - \int_a^b g = 0$, Exercise 7.2-8 implies that $f - g = 0$ on $[a, b]$, which contradicts the assumption at the beginning. □

7.2-18 Let f be continuous on $[a, b]$, let $f(x) \geq 0$ for $x \in [a, b]$, and let $M_n := (\int_a^b f^n)^{1/n}$. Show that $\lim(M_n) = \sup\{f(x) : x \in [a, b]\}$.

Solution. Denote $\|f\|_\infty = \sup\{f(x) : x \in [a, b]\}$. Without loss of generality, we may assume that $\|f\|_\infty > 0$.

Let $0 < \varepsilon < \|f\|_\infty$. By definition of supremum, there is $x_0 \in [a, b]$ such that $f(x_0) > \|f\|_\infty - \varepsilon/2$. Since f is continuous at x_0, there is a subinterval $[c, d] \subseteq [a, b]$ such that

$$f(x) > f(x_0) - \varepsilon/2 \geq \|f\|_\infty - \varepsilon > 0 \quad \text{for any } x \in [c, d].$$

Now

$$\int_a^b f^n \geq \int_c^d f^n \geq \int_c^d (\|f\|_\infty - \varepsilon)^n = (d - c)(\|f\|_\infty - \varepsilon)^n.$$

And thus,

$$(d - c)^{1/n}(\|f\|_\infty - \varepsilon) \leq M_n = \left(\int_a^b f^n\right)^{1/n} \leq (b - a)^{1/n}\|f\|_\infty.$$

Passing $n \to \infty$ yields

$$\|f\|_\infty - \varepsilon \leq \liminf_n M_n \leq \limsup_n M_n \leq \|f\|_\infty.$$

Since $\varepsilon > 0$ can be arbitrarily small, we have $\liminf(M_n) = \limsup(M_n) = \|f\|_\infty$, that is

$$\lim(M_n) = \|f\|_\infty = \sup\{f(x) : x \in [a, b]\}.$$

$\square$