MATH 2068 Mathematical Analysis I
2023-24 Term 2
Suggested Solution to Homework 3

6.4-10 Let $h(x) := e^{-1/x^2}$ for $x \neq 0$ and $h(0) := 0$. Show that $h^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Conclude that the remainder term in Taylor's Theorem for $x_0 = 0$ does not converge to zero as $n \to \infty$ for $x \neq 0$.

Solution. First, we show that $\lim_{x\to 0} h(x)/x^k = 0$ for any $k \in \mathbb{N}$. By successive application of L'Hospital's Rule,

$$\lim_{y\to+\infty} \frac{y^k}{e^y} = \lim_{y\to+\infty} \frac{ky^{k-1}}{e^y} = \cdots = \lim_{y\to+\infty} \frac{k!}{e^y} = 0 \quad \text{for any } k \in \mathbb{N}.$$

Let $y = 1/x^2$. Then $y \to +\infty$ as $x \to 0$. Hence, for any $k \in \mathbb{N}$,

$$\lim_{x\to 0} \frac{h(x)}{x^k} = \lim_{x\to 0} \frac{(1/x^2)^k}{e^{1/x^2}} \cdot x^k = 0. \qquad (*)$$

Next, we calculate $h^{(n)}(x)$ for $x \neq 0$. Clearly $h(x) = e^{-1/x^2}$ is infinitely differentiable for $x \neq 0$. By applying Leibniz's rule to $h'(x) = \frac{2}{x^3} e^{-1/x^2} = \frac{2}{x^3} h(x)$, we have

$$h^{(n+1)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2}{x^3}\right)^{(n-k)} h^{(k)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{(n-k+2)!}{x^{n-k+3}} h^{(k)}(x) \qquad (**)$$

for any $x \neq 0$ and integer $n \geq 0$.

Now, we prove by induction on n that

(i) $\lim_{x\to 0} \dfrac{h^{(n)}(x)}{x^m}$ for any $m \in \mathbb{N}$;

(ii) $h^{(n)}(0) = 0$.

The case $n = 0$ follows immediately from $(*)$. Suppose (i) and (ii) are true for n. Then $(**)$ gives

$$\lim_{x\to 0} \frac{h^{(n+1)}(x)}{x^m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (n-k+2)! \left(\lim_{x\to 0} \frac{h^{(k)}(x)}{x^{n-k+3+m}} \right) = 0.$$

Moreover,

$$h^{(n+1)}(0) = \lim_{x\to 0} \frac{h^{(n)}(x) - h^{(n)}(0)}{x - 0} = \lim_{x\to 0} \frac{h^{(n)}(x)}{x} = 0.$$

This completes the induction.

Finally, the remainder term in Taylor's Theorem is given by

$$R_n(x) = h(x) - \sum_{k=0}^{n} \frac{h^{(k)}(0)}{k!} x^k = h(x),$$

and so $\lim_{x\to 0} R_{n+1}(x) = h(x) \neq 0$ for $x \neq 0$. $\square$

7.1-15 If $f \in \mathcal{R}[a,b]$ and $c \in \mathbb{R}$, we define g on $[a+c, b+c]$ by $g(y) := f(y-c)$. Prove that $g \in \mathcal{R}[a+c, b+c]$ and that $\int_{a+c}^{b+c} g = \int_a^b f$. The function g is called the c-**translate** of f.

Solution. If $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a tagged partition of $[a+c, b+c]$, we define a tagged partition of $[a,b]$ by $\dot{\mathcal{P}}_c := \{([x_{i-1} - c, x_i - c], t_i - c)\}_{i=1}^n$. Clearly $\|\dot{\mathcal{P}}_c\| = \|\dot{\mathcal{P}}\|$. Moreover,

$$S(g; \dot{\mathcal{P}}) = \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) = \sum_{i=1}^n f(t_i - c)((x_i - c) - (x_{i-1} - c)) = S(f; \dot{\mathcal{P}}_c).$$

Let $\varepsilon > 0$. Since $f \in \mathcal{R}[a,b]$, there exists $\delta > 0$ such that if $\dot{\mathcal{Q}}$ is any tagged partition of $[a,b]$ with $\|\dot{\mathcal{Q}}\| < \delta$, then

$$\left| S(f; \dot{\mathcal{Q}}) - \int_a^b f \right| < \varepsilon.$$

Now, if $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a tagged partition of $[a+c, b+c]$ with $\|\dot{\mathcal{P}}\| < \delta$, then $\dot{\mathcal{P}}_c$ is a tagged partition of $[a,b]$ with $\|\dot{\mathcal{P}}_c\| = \|\dot{\mathcal{P}}\| < \delta$, and that

$$\left| S(g; \dot{\mathcal{P}}) - \int_a^b f \right| = \left| S(f; \dot{\mathcal{P}}_c) - \int_a^b f \right| < \varepsilon.$$

Therefore, $g \in \mathcal{R}[a+c, b+c]$ and $\displaystyle\int_{a+c}^{b+c} g = \int_a^b f$. □