MATH 2068 Mathematical Analysis I
2023-24 Term 2
Suggested Solution to Homework 2

6.2-13 Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Show that if f' is positive on I, then f is strictly increasing on I.

Solution. Let $a, b \in I$ such that $a < b$. Since f is differentiable on I, f is continuous on $[a, b]$ and differentiable on (a, b). By Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

By assumption f' is positive on I, so the above gives $f(b) > f(a)$. As $a < b$ are arbitrary points in I, we conclude that f is strictly increasing on I.

□

6.2-16 Let $f : [0, \infty) \to \mathbb{R}$ be differentiable on $(0, \infty)$ and assume that $f'(x) \to b$ as $x \to \infty$.

(a) Show that for any $h > 0$, we have $\lim_{x\to\infty} (f(x + h) - f(x))/h = b$.

(b) Show that if $f(x) \to a$ as $x \to \infty$, then $b = 0$.

(c) Show that $\lim_{x\to\infty} (f(x)/x) = b$.

Solution. (a) Let $\varepsilon > 0$. Since $f'(x) \to b$ as $x \to \infty$, there exists $c > 0$ such that

$$|f'(x) - b| < \varepsilon/3 \qquad \text{whenever } x > c.$$

Suppose $x > c$ and $h > 0$. Since f is continuous on $[x, x+h]$ and differentiable on $(x, x+h)$, there exists $\xi \in (x, x + h)$ such that

$$\frac{f(x + h) - f(x)}{h} = f'(\xi).$$

As $\xi > x > c$, we have

$$\left|\frac{f(x + h) - f(x)}{h} - b\right| = \left|f'(\xi) - b\right| < \varepsilon.$$

Therefore $\lim_{x\to\infty} (f(x + h) - f(x))/h = b$.

(b) The assumption implies that, for any $h > 0$, $\lim_{x\to\infty} f(x + h) = \lim_{x\to\infty} f(x) = a$. Hence

$$b = \lim_{x\to\infty} \frac{f(x + h) - f(x)}{h} = \frac{a - a}{h} = 0.$$

(c) Let $\varepsilon > 0$. Since $\lim_{x\to\infty} f'(x) = b$, there is $c > 0$ such that

$$|f'(x) - b| < \varepsilon/3 \qquad \text{whenever } x > c.$$

By Mean Value Theorem, for any $x > c$, there is $\xi_x \in (c, x)$ such that

$$f(x) - f(c) = f'(\xi_x)(x - c).$$

Thus

$$\frac{f(x)}{x} - b = (f'(\xi_x) - b)(1 - \frac{c}{x}) - \frac{bc}{x} + \frac{f(c)}{x}.$$

Let $M = \max\{|f(c)|, c(|b|+1)\}$. Now if $x > \max\{3M/\varepsilon, c\}$, then $\xi_x > c$ and hence

$$\begin{aligned}\left|\frac{f(x)}{x} - b\right| &\leq |f'(\xi_x) - b|(1 - \frac{c}{x}) + \frac{c|b|}{x} + \frac{|f(c)|}{x} \\ &< \varepsilon/3 \cdot 1 + M \cdot \frac{\varepsilon}{3M} + M \cdot \frac{\varepsilon}{3M} \\ &= \varepsilon.\end{aligned}$$

Therefore $\lim_{x\to\infty} \frac{f(x)}{x} = b$.

□

6.4-4 Show that if $x > 0$, then $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$.

Solution. Let $f(x) = \sqrt{1+x}$. Then, for any $x > -1$,

$$f'(x) = \frac{1}{2\sqrt{1+x}}, \quad f''(x) = -\frac{1}{4(1+x)^{3/2}}, \quad f'''(x) = \frac{3}{8(1+x)^{5/2}}.$$

Fix $x > 0$. By Taylor's Theorem, there exists $c_1 \in (0, x)$ such that

$$\begin{aligned}f(x) &= f(0) + f'(0)(x-0) + \frac{f''(c_1)}{2!}(x-0)^2 \\ &= 1 + \frac{1}{2}x - \frac{1}{8(1+c_1)^{3/2}}x^2.\end{aligned}$$

Since $-\frac{1}{8(1+c_1)^{3/2}}x^2 < 0$, we have $\sqrt{1+x} \leq 1 + \frac{1}{2}x$.

Similarly, there exists $c_2 \in (0, x)$ such that

$$\begin{aligned}f(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(c_2)}{3!}(x-0)^3 \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16(1+c_2)^{5/2}}x^3.\end{aligned}$$

Since $\frac{1}{16(1+c_2)^{5/2}}x^3 > 0$, we have $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x}$. □