MATH 2068 Mathematical Analysis I
2023-24 Term 2
Suggested Solution to Homework 1

6.1-12 If $r > 0$ is a rational number, let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^r \sin(1/x)$ for $x \neq 0$, and $f(0) := 0$. Determine those values of r for which $f'(0)$ exists.

Solution. Since x^r may not be defined for $x < 0$, we assume that $f(x) = |x|^r \sin(1/x)$ for $x \neq 0$. For $x \neq 0$, we have

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|^r \sin(1/x)}{x},$$

and hence

$$\left| \frac{f(x) - f(0)}{x - 0} \right| = |x|^{r-1} |\sin(1/x)| \leq |x|^{r-1}.$$

- If $r > 1$, then $\lim_{x \to 0} |x|^{r-1} = 0$, and it follows from the squeeze theorem that

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0.$$

- If $0 < r \leq 1$, consider two positive sequences (x_n) and (y_n) defined by

$$x_n = \frac{1}{2n\pi} \quad \text{and} \quad y_n = \frac{1}{2n\pi + \pi/2}.$$

  Clearly, $\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 0$. However,

$$\lim_{n \to \infty} \frac{f(x_n) - f(0)}{x_n - 0} = \lim_{n \to \infty} 0 = 0,$$

$$\lim_{n \to \infty} \frac{f(y_n) - f(0)}{y_n - 0} = \lim_{n \to \infty} (2n\pi + \frac{\pi}{2})^{1-r} = \begin{cases} \infty & \text{if } 0 < r < 1 \\ 1 & \text{if } r = 1. \end{cases}$$

  By sequential criterion, $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist.

In conclusion, $f'(0)$ exists if and only if $r > 1$. □

6.1-15 Given that the restriction of the cosine function cos to $I := [0, \pi]$ is strictly decreasing and that $\cos 0 = 1$, $\cos \pi = 1$, let $J := [-1, 1]$, and let $\operatorname{Arccos} : J \to \mathbb{R}$ be the function inverse to the restriction of cos to I. Show that Arccos is differentiable on $(-1, 1)$ and $D \operatorname{Arccos} y = (-1)/(1 - y^2)^{1/2}$ for $y \in (-1, 1)$. Show that Arccos is not differentiable at -1 and 1.

Solution. Since Arccos is the function inverse to the restriction of cos to I, we have if $x \in [0, \pi]$ and $y \in [-1, 1]$, then

$$y = \cos x \quad \text{if and only if} \quad \operatorname{Arccos} y = x.$$

It was asserted (without proof) in Example 6.1.7(d) that cos is differentiable on I and that $D \cos x = -\sin x$ for $x \in I$. Since $\sin x \neq 0$ for $x \in (0, \pi)$, it follows from Theorem 6.1.8 that

Arccos is differentiable on $(-1,1)$ and

$$\begin{aligned} D\,\mathrm{Arccos}\, y &= \frac{1}{D\cos x} = \frac{1}{-\sin x} \\ &= \frac{1}{-\sqrt{1-(\cos x)^2}} = \frac{-1}{\sqrt{1-y^2}} \end{aligned}$$

for all $y \in (-1,1)$. Arccos is not differentiable at -1 and 1 since $D\cos$ is zero at $x = 0, \pi$ and $\cos 0 = 1$, $\cos \pi = -1$.

□

6.2-7 Use the Mean Value Theorem to prove that $(x-1)/x < \ln x < x - 1$ for $x > 1$.

Solution. Fix $x > 1$ and define $f : [1, x] \to \mathbb{R}$ by $f(t) := \ln t$. Since f is continuous on $[1, x]$ and differentiable on $(1, x)$, the Mean Value Theorem implies that there is $c \in (1, x)$ such that

$$f(x) - f(1) = f'(c)(x-1).$$

Since $f'(t) = 1/t$ for $t \in (1, x)$ and $1 < c < x$, we have

$$1/x < f'(c) = 1/c < 1.$$

Noting that $f(1) = 0$ and $x - 1 > 0$, we have

$$(x-1)/x < \ln x < x - 1.$$

□