Pf of Thm 9.4.10: Since $[a,b] \subset (-R,R)$, $\exists\ 0<C<1$ such that

$-CR<a$ and $b<CR$. (Note: C depends only on a,b)

Therefore $\forall\ x \in [a,b]$, $|x| < CR$.

By argument in the proof of Cauchy-Hadamard Thm, we have

$\exists\ K \in \mathbb{N}$ s.t. $|a_n x^n| \le C^n$, $\forall n \ge K$ (Ex! use $0<C_1<C$ s.t. $-C_1R<a<b<C_1R$ to find a K indep. of x)

Since $\sum C^n$ is convergent, Weierstrass M-Test (Thm 9.4.6)

$\Rightarrow$ ($\sum_{n=K}^{\infty} a_n x^n$ and hence) $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[a,b]$. $\#$

Thm 9.4.11

- The limit of power series is continuous on the interval of convergence.
- A power series can be integrated term-by-term over any closed and bounded interval contained in the interval of convergence.

Pf: • $\forall\ x \in (-R,R)$, choose a closed & bounded interval $[a,b]$ s.t. $x \in [a,b] \subset (-R,R)$. Then on $[a,b]$,

$\sum a_n x^n$ converges uniformly. (Thm 9.4.10)

Thm 9.4.2 $\Rightarrow$ $\sum_{n=1}^{\infty} a_n x^n$ is continuous on $[a,b]$ and hence at x

Since $x \in (-R,R)$ is arbitrary, $\sum_{n=0}^{\infty} a_n x^n$ is continuous on $(-R,R)$.

- For any closed and bounded interval $[a,b] \subset (-R,R)$,

  $\sum a_n x^n$ converges uniformly on $[a,b]$

  and hence Thm 9.4.3 $\Rightarrow$ integrability and

$$\int_a^b \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \int_a^b a_n x^n . \quad \divideontimes$$

Thm 9.4.12 (Differentiation Thm)

A power series can be differentiated term-by-term within the interval of convergence. In fact, if R = radius of convergence of $\sum a_n x^n$

and $\quad f(x) = \sum_{n=0}^{\infty} a_n x^n$, for $|x| < R$,

then the radius of convergence of $\sum_{n=0}^{\infty} n a_n x^{n-1} = R$,

and $\quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, for $|x| < R$

Pf: All conditions of Thm 8.2.3 on interchanging of limit and derivative are clearly satisfied when restricted to a closed and bounded interval $[a,b] \subset (-R,R)$ (using Thm 9.4.10) except the uniform convergence of the $\sum (a_n x^n)' = \sum n a_n x^{n-1}$ on $[a,b]$ needs a proof.

By Thm 9.4.10, we only need to prove the following

Radius of convergence of $\sum n a_n x^{n-1} = R$

$\qquad\qquad$ = Radius of convergence of $\sum a_n x^n$

Pf

Since $n^{\frac{1}{n}} \to 1$, the seq. $\left(|(n+1)a_{n+1}|^{\frac{1}{n}}\right)$ is bounded

$\qquad\qquad \Leftrightarrow$ the seq $\left(|a_n|^{\frac{1}{n}}\right)$ is bounded

unbounded case:

$R=0 \Leftrightarrow$ Radius of convergence of $\sum n a_n x^{n-1} = 0$

bounded case:

Radius of convergence of $\sum n a_n x^{n-1} = \limsup |(n+1) a_{n+1}|^{\frac{1}{n+1}}$ (check!)

$$= \limsup |n a_n|^{\frac{1}{n}} = \limsup \left(n^{\frac{1}{n}} |a_n|^{\frac{1}{n}}\right)$$

$$= \limsup |a_n|^{\frac{1}{n}} \quad (\text{since } n^{\frac{1}{n}} \to 1)$$

$$= R.$$

$\therefore$ The claim and hence the Thm is proved since $[a,b] \subset (-R,R)$ is arbitrary.

$\#$

Remarks: (i) Differentiation Thm 9.4.12 makes no conclusion for $|x|=R$:

e.g. $\sum \frac{1}{n^2}x^n$ converges for $|x|=1$ $(=R)$

but $\left(\sum \frac{1}{n^2}x^n\right)' = \sum \frac{1}{n}x^{n-1}$ $\begin{cases} \text{converges at } x=-1 \\ \text{diverges at } x=1. \end{cases}$

(ii) Repeated application of Thm 9.4.12 $\Rightarrow$

$\forall k\in\mathbb{N}$, $\boxed{\left(\sum_{n=0}^{\infty} a_n x^n\right)^{(k)} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}}$ $(|x|<R)$

Thm 9.4.13 (Uniqueness Thm)

If $\sum a_n x^n$ & $\sum b_n x^n$ converge to the same function f on an interval $(-r,r)$, $r>0$, then

$$a_n = b_n, \quad \forall n\in\mathbb{N}$$

(In fact $a_n = b_n = \frac{1}{n!} f^{(n)}(0)$)

Pf: By remark (ii) of Thm 9.4.12, $\forall k\in\mathbb{N}$,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} \quad \forall x\in(-r,r).$$

$$\Rightarrow f^{(k)}(0) = \frac{k!}{(k-k)!} a_k \quad (0^{n-k}=0 \text{ for } n>k)$$

$$\Rightarrow a_k = \frac{1}{k!} f^{(k)}(0) \qquad \text{Same for } b_k. \quad ※$$

Taylor Series

Let f has derivatives of all orders at a point $c \in \mathbb{R}$,

then one can form a power series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

Note that $\begin{cases} \bullet \text{ no convergence yet (unless } x=c) \\ \bullet \text{ Even it converges, it may \underline{not} equal } f \text{ (Ex. 9.4.12)} \end{cases}$

<u>Def</u> We say that $\displaystyle f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

is the <u>Taylor expansion</u> of f at c if $\exists R>0$ such that

$\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ converges to $f(x)$ on $(c-R, c+R)$,

and $\dfrac{f^{(n)}(c)}{n!}$ are called <u>Taylor coefficients</u>.

(i.e. The remainder $R_n(x)$ in Taylor's Thm $\to 0$ on $(c-R, c+R)$)

<u>Remark</u>: By Uniqueness Thm 9.4.13, if Taylor expansion exists, it is unique.

Eg 9.4.14

(a) $f(x) = \sin x$, $x \in \mathbb{R}$,

then $f^{(n)}(x) = \begin{cases} (-1)^k \sin x, & \text{if } n=2k \\ (-1)^k \cos x, & \text{if } n=2k+1. \end{cases}$

At $c=0$, we have $f^{(n)}(0) = \begin{cases} 0, & \text{if } n=2k \\ (-1)^k, & \text{if } n=2k+1 \end{cases}$

Furthermore, by Taylor's Thm 6.4.1, the remainder $R_n(x)$ satisfies

$$|R_n(x)| = \frac{|f^{(n+1)}(c_1)|\,|x|^{n+1}}{(n+1)!} \quad \text{for some } c_1 \text{ between } x \,\&\, 0$$

$$\le \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0$$

$$\therefore \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \forall x \in \mathbb{R}$$

is the Taylor expansion of $\sin x$ at $x=0$.

Then application of Differentiation Thm 9.4.12, we have

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad \forall x \in \mathbb{R}$$

is the Taylor expansion of $\cos x$ at $x=0$.

Remarks: (i) In this example, we used "Remainder of Taylor's series" to calculate the radius of convergence, not directly from definition or using $\frac{1}{\rho} = \lim \frac{|a_n|}{|a_{n+1}|}$ (when limit exists)

Note that the series only have "even" terms or "odd" terms,

$a_{2k+1} = 0$ or $a_{2k} = 0$. Hence $\frac{|a_n|}{|a_{n+1}|}$ is not well-defined and hence $\lim \frac{|a_n|}{|a_{n+1}|}$ cannot be used.

To use definition $\rho = \limsup |a_n|^{\frac{1}{n}}$, we note for eg:

that for sine series: $a_n = \begin{cases} \frac{(-1)^k}{(2k+1)!} & \text{if } n = 2k+1 \\ 0 & \text{if } n = 2k \end{cases}$

$\therefore$ the seq. $|a_n|^{\frac{1}{n}} = \left(\left(\frac{1}{3!}\right)^{\frac{1}{3}}, 0, \left(\frac{1}{5!}\right)^{\frac{1}{5}}, 0, \cdots \right)$ doesn't converge,

but $\limsup |a_n|^{\frac{1}{n}} = \lim_{k \to \infty} \left[\frac{1}{(2k+1)!}\right]^{\frac{1}{2k+1}} = 0 \quad \therefore R = +\infty$.

(ii) But calculation of radius of convergence doesn't prove the Taylor's series converges to the "original function".

(b) $g(x) = e^x, \ x \in \mathbb{R}$

Then $g^{(n)}(x) = e^x, \ \forall x \in \mathbb{R} \Rightarrow g^{(n)}(0) = 1$.

By Taylor's Thm 6.4.1, the remainder satisfies

$|R_n(x)| \leq \frac{e^c}{(n+1)!} |x|^{n+1}$ for some c between x & 0.

$\leq \frac{e^{|x|} |x|^{n+1}}{(n+1)!} \longrightarrow 0$ as $n \to \infty$.

$$\therefore \quad e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad , \forall x \in \mathbb{R}$$

is the Taylor expansion of e^x at $x=0$.

Furthermore, by $e^x = e^c e^{x-c} = e^c \sum_{n=0}^{\infty} \frac{1}{n!}(x-c)^n$,

we see that $e^x = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n$ is the

Taylor expansion of e^x at $x=c$. ※

<u>Remarks</u>: (i) This implies the radius of convergence $= +\infty$ (says at $c=0$).

Of course, one can derive it from calculating

$$\left(\frac{1}{n!}\right)^{\frac{1}{n}} \to 0 \quad \text{as } n \to +\infty.$$

Since $|a_n|^{\frac{1}{n}} = \left(\frac{1}{n!}\right)^{\frac{1}{n}}$ and limit exists,

$$\therefore \rho = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{n!}\right)^{\frac{1}{n}} = 0 \Rightarrow R = +\infty.$$

(ii) The radius of convergence R can be calculated by

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n\to\infty} (n+1) = \infty.$$

Review

Ch6 Differentiation

§6.1 Derivative (Chain rule, Inverse function)

§6.2 Mean value Thm (Rolle's Thm, 1^{st} derivative test for Extrema)

§6.3 L'Hospital's Rules

§6.4 Taylor's Thm (derivative form of remainder,
relative extrema, convex function, Newton's method)

Ch7 Riemann Integral

§7.1 Riemann integral (partition, tagged partition, Riemann sum,
Riemann integrable, boundedness thm)

§7.2 Riemann integrable functions (Cauchy Criterion,
Squeeze Thm, "classes" of Riemann integrable functions,
additivity Thm)

(Midterm up to here)

§7.3 The Fundamental Thm (1^{st} form $\int_a^b f = F(b) - F(a)$,
2^{nd} form $\frac{d}{dx}\int_a^x f = f(x)$; substitution Thm,
Lebesgue's Integrability Criterion (pf omitted), Integration by Parts,
Taylor's Thm with integral form remainder)

§7.4 The Darboux Integral (Upper & lower sums, upper & lower integrals, integrability criterion, equivalence to Riemann integral)

(§7.5 Omitted)

Ch8 Sequences of Functions

§8.1 Pointwise & Uniform Convergence (uniform norm, Cauchy Criterion)

§8.2 Interchange of Limits (Limit & Continuity, Limit & Derivatives, Limit & Integral, Dini's Thm)

§8.3 Exponential & Logarithmic Functions (Definitions & basic properties)

§8.4 Trigonometric Functions (Definitions & basic properties)

Ch9 Infinite Series

§9.1 Absolute Convergence (conditional convergence, grouping, rearrangement)

§9.2 Tests for Absolute Convergence (Comparison Test, Root Test, Ratio Test, and their limit version, Integral Test, Raabe's Test)

§9.3 Tests for Nonabsolute Convergence (alternating series, Abel's Test, Dirichlet Test)

§9.4 Series of Functions (pointwise & uniform convergences, Cauchy Criterion for Uniform convergence, M-Test, Power Series: radius of convergence, uniform convergence when restrict closed & bdd subinterval, continuity, differentiation & integration term-by-term)

(End)

Final exam: $\boxed{\text{May 8 (Wednesday) 3:30–5:30 pm, U Gym}}$

(5 questions as in Mid-term)

covers all material including those in lectures, tutorials, homework, & textbook (including all exercises in Textbook no matter it's assigned in homework or not) with emphasis on those material after mid-term (i.e. §7.3 – §9.4).

But those material before mid-term (i.e. §6.1 – §7.2) may also be tested directly/explicitly or indirectly/implicitly.