Pf (of Thm 9.3.2): Consider partial sum

$$S_{2n} = \sum_{k=1}^{2n} (-1)^{k+1} z_k = z_1 - z_2 + z_3 - z_4 + \cdots + z_{2n-1} - z_{2n}$$

Then $S_{2(n+1)} - S_{2n} = z_{2n+1} - z_{2n+2} \geq 0$, since z_n is decreasing

$\therefore$ (S_{2n}) is increasing (in n).

Also $z_1 - S_{2n} = \underbrace{z_2 - z_3}_{\geq 0} + \underbrace{z_4 - z_5}_{\geq 0} + \cdots + \underbrace{z_{2n-2} - z_{2n-1}}_{\geq 0} + \underbrace{z_{2n}}_{> 0} > 0$

$\therefore$ (S_{2n}) is bounded above by z_1

By Monotone Convergence Thm (Thm 3.3.2), $\exists\, S \in \mathbb{R}$ s.t.

$$S_{2n} \to S \quad \text{as } n \to \infty.$$

Together with $z_n \to 0$ as $n \to \infty$, we have

$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t.

if $n \geq K$, then $\begin{cases} \bullet\ |S_{2n} - S| < \frac{\varepsilon}{2}, \text{ and} \\ \bullet\ (0<)\ z_{2n+1} < \frac{\varepsilon}{2}. \end{cases}$

Then $|S_{2n+1} - S| = |z_{2n+1} + S_{2n} - S|$

$$\leq |z_{2n+1}| + |S_{2n} - S| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore$ $S_{2n+1} \to S$ as $n \to \infty$.

Combining with $S_{2n} \to S$ as $n \to \infty$, we have

$$\lim S_n = S$$

$\therefore$ $\sum (-1)^{n+1} z_n$ is convergent. ✕

The Dirichlet and Abel Tests

Thm 9.3.3 (Abel's Lemma)

Let • $(x_n), (y_n)$ be sequences in $\mathbb{R}$, and

• $\begin{cases} S_0 = 0, & \& \\ S_n = \sum_{k=1}^{n} y_k, & n = 1, 2, 3, \ldots \end{cases}$

Then for $m > n$,

$$\sum_{k=n+1}^{m} x_k y_k = (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k.$$

Pf:
$$\sum_{k=n+1}^{m} x_k y_k = \sum_{k=n+1}^{m} x_k (S_k - S_{k-1})$$

$$= x_m (S_m - S_{m-1}) + x_{m-1}(S_{m-1} - S_{m-2}) + \cdots + x_{n+1}(S_{n+1} - S_n)$$

$$= x_m S_m - (x_m - x_{m-1}) S_{m-1} - (x_{m-1} - x_{m-2}) S_{m-2} - \cdots$$
$$- (x_{n+2} - x_{n+1}) S_{n+1} - x_{n+1} S_n$$

$$= (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k$$

#

Thm 9.3.4 (Dirichlet's Test)

If $\begin{cases} \bullet\ (x_n) \text{ decreasing } \& \lim x_n = 0 \\ \bullet\ (S_n = \sum_{k=1}^{n} y_k) \text{ are bounded}, \end{cases}$

then $\sum x_n y_n$ is convergent.

Pf: (s_n) bdd $\Rightarrow \exists B>0$ s.t. $|s_n| \le B$, $\forall n \in \mathbb{N}$.

Then Abel's Lemma (Thm 9.3.3) $\Rightarrow$ for $m>n$,

$$\left|\sum_{k=n+1}^{m} x_k y_k\right| \le |x_m s_m - x_{n+1} s_n| + \sum_{k=n+1}^{m-1} |x_k - x_{k+1}||s_k|$$

$$\le (x_m + x_{n+1})B + \sum_{k=n+1}^{m-1} (x_k - x_{k+1})B$$

$$= B[(x_m + x_{n+1}) + (x_{n+1} - x_m)] \qquad \left(\begin{array}{l}\text{since } x_n \text{ decreasing} \to 0 \\ x_k - x_{k+1} \ge 0, \ \& \ x_n \ge 0\end{array}\right)$$

$$= 2x_{n+1}B \longrightarrow 0 \quad \text{as } n \to \infty$$

$\therefore \forall \varepsilon>0$, $\exists K \in \mathbb{N}$ s.t. if $m>n\ge K$, $\left|\sum_{k=n+1}^{m} x_n y_n\right| < \varepsilon$

By Cauchy Criterion (Thm 3.7.4), $\sum x_n y_n$ is convergent. #

Thm 9.3.5 (Abel's Test)

If
- (x_n) convergent monotone sequence
- $\sum y_n$ convergent

Then $\sum x_n y_n$ is also convergent.

(multiplying convergent monotone coefficients to a convergent series results in a convergent series.)

Pf: <u>Case 1</u>: (x_n) decreasing & $\lim x_n = x$

Let $u_n = x_n - x$, $\forall n \in \mathbb{N}$.

Then (u_n) decreasing & $u_n \to 0$.

Now $\sum y_n$ converges $\Rightarrow$ partial sum of $\sum y_n$ are bounded

$\therefore$ Dirichlet's Test (Thm 9.3.4) $\Rightarrow$ $\sum u_n y_n$ is convergent.

Hence $\sum x_n y_n = \sum (u_n + x) y_n = \sum u_n y_n + x \sum y_n$

is also convergent.

<u>Case 2</u> (x_n) increasing, $x = \lim x_n$.

Similar argument as in case 1 by considering

$v_n = x - x_n$, $\forall n \in \mathbb{N}$ instead of u_n. (Ex!)

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<u>Eg 9.3.6</u>

(a) Recall $2\left(\sin\frac{1}{2}x\right)(\cos x + \cdots + \cos nx) = \sin(n+\frac{1}{2})x - \sin\frac{1}{2}x$ (Ex)

If x is fixed and $x \neq 2k\pi$, $\forall k = \cdots -2, -1, 0, 1, 2, \cdots$

Then $\underbrace{|\cos x + \cdots + \cos nx|}_{\text{partial sum of } \sum \cos nx} = \dfrac{|\sin(n+\frac{1}{2})x - \sin\frac{1}{2}x|}{2|\sin\frac{1}{2}x|} \le \underbrace{\dfrac{1}{|\sin\frac{1}{2}x|}}_{\text{bounded indep. of } n}$, $\forall n \in \mathbb{N}$

$\therefore$ For a fixed $x \neq 2k\pi$, Dirichlet's Test $\Rightarrow$

$\sum_{n=1}^{\infty} a_n \cos nx$ converges, provided $\begin{cases} (a_n) \text{ is decreasing } \& \\ \lim a_n = 0. \end{cases}$

(b) Similarly, from

$$2(\sin\tfrac{1}{2}x)(\sin x + \cdots + \sin nx) = \cos\tfrac{1}{2}x - \cos(n+\tfrac{1}{2})x \quad , \forall n \in \mathbb{N}$$

we have, for $x \neq 2k\pi$,

$$|\sin x + \cdots + \sin nx| \leq \frac{1}{|\sin\frac{1}{2}x|} \qquad \forall n \in \mathbb{N}$$

partial sum of $\sum \sin nx$ — bounded indep. of n

$$\Rightarrow \quad \sum_{n=1}^{\infty} a_n \sin nx \text{ converges for } x \neq 2k\pi$$

provided $\begin{cases} (a_n) \text{ decreasing and} \\ \lim a_n = 0 . \end{cases}$

§9.4 Series of Functions

Def 9.4.1

If (f_n) is a seq. of functions defined on $D \subseteq \mathbb{R}$ (with $\mathbb{R}$-value), then the sequence of partial sums (S_n) of the infinity series of functions $\sum f_n$ is defined by

$$S_n(x) = \sum_{k=1}^{n} f_k(x), \quad \forall x \in D$$

- If (S_n) converges to a function f on D, then we say that the infinite series of functions $\sum f_n$ converges to f on D.

(usually write $f(x) = \sum_{n=1}^{\infty} f_n(x)$, $f = \sum_{n=1}^{\infty} f_n$, or $f = \sum f_n$)

- If $\sum |f_n(x)|$ converges $\forall x \in D$, then we say that $\sum f_n$ is absolutely convergent on D.
- If $S_n \rightrightarrows f$ (uniformly) on D, then we say that $\sum f_n$ is uniformly convergent on D, or $\sum f_n$ converges to f uniformly on D

Using $S_n \rightrightarrows f \Leftrightarrow \sum f_n$ converges to f uniformly, Thm 8.2.2, Thm 8.2.3 & Thm 8.2.4 imply the following theorems immediately:

Thm 9.4.2 If
- f_n continuous on D, $\forall n \in \mathbb{N}$
- $\sum f_n$ converges to f uniformly on D

Then f is continuous on D

(Pf : Applying Thm 8.2.2 to $S_n \rightrightarrows f$.)

Thm 9.4.3 If
- $f_n \in R[a,b]$, $\forall n \in \mathbb{N}$ $(a<b \in \mathbb{R})$
- $\sum f_n$ converges to f uniformly on $[a,b]$

Then $f \in R[a,b]$ and $\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$

$$\left(\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n \right)$$

(Pf : Applying Thm 8.2.4 to $S_n \rightrightarrows f$.)

Thm 9.4.4 If
- $f_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$ $(a<b \in \mathbb{R})$,
- f_n' exists on $[a,b]$, $\forall n \in \mathbb{N}$,
- $\exists x_0 \in [a,b]$ s.t. $\sum f_n(x_0)$ converges,
- $\sum f_n'$ converges uniformly on $[a,b]$.

Then $\exists f : [a,b] \to \mathbb{R}$ such that
- $\sum f_n$ converges to f uniformly on $[a,b]$,
- f' exists and $f' = \sum_{n=1}^{\infty} f_n'$

(Pf : Applying Thm 8.2.3 to S_n with S_n' converges uniformly etc.)

Tests for Uniform Convergence

Thm 9.4.5 (Cauchy Criterion)

$\sum f_n$ is uniformly convergent on D $\iff$

$\forall \varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ such that

if $m > n \geq K(\varepsilon)$, then $|f_{n+1}(x) + \cdots + f_m(x)| < \varepsilon$, $\forall x \in D$.

(Pf: Applying Cauchy Criterion for Uniform Convergence (Thm 8.1.10)

to S_n and observing that

$$S_m(x) - S_n(x) = f_{n+1}(x) + \cdots + f_m(x).\ )$$

Thm 9.4.6 (Weierstrass M-Test)

If $\begin{cases} \bullet\ |f_n(x)| \leq M_n, \ \forall x \in D, \ \forall n \in \mathbb{N} \\ \bullet\ \sum M_n \text{ is convergent} \end{cases}$

then $\sum f_n$ is uniformly convergent on D

Pf: $\sum M_n$ convergent & $M_n \geq 0 \Rightarrow$

$\forall \varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ such that

if $m > n \geq K(\varepsilon)$, then $M_{n+1} + \cdots + M_m < \varepsilon$. (Thm 3.7.4)

Hence $|f_{n+1}(x) + \cdots + f_m(x)| \leq M_{n+1} + \cdots + M_m < \varepsilon$.

Cauchy Criterion (Thm 9.4.5) $\Rightarrow$ $\sum f_n$ converges uniformly on D ⨳

Power Series

Def 9.4.7 If $f_n(x) = a_n(x-c)^n$, a_n & $c \in \mathbb{R}$, $\forall n = 0,1,2,\cdots$

then $\sum f_n(x) = \sum a_n(x-c)^n$

is called a <u>power series around $x=c$</u>.

<u>Remarks</u>: • Power series usually starts with $n=0$ (instead of $n=1$):

$$\sum a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

• $\sum a_n x^n$ may <u>not</u> defined over all of $\mathbb{R}$:

(i) $\sum_{n=0}^{\infty} n!\, x^n$ converges only for $x=0$, (Ex!)

(ii) $\sum_{n=0}^{\infty} x^n$ converges for $|x|<1$, (geometric series)

(iii) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges $\forall x \in \mathbb{R}$, (exponential function)

Hence there is a need to determine

the <u>set</u> on which $\sum a_n x^n$ converges.

In the following, we consider the case that "$c=0$".

This is no loss of generality as the "translation $y=x-c$"

redues $\sum a_n(x-c)^n$ to $\sum a_n y^n$.

Recall: (Def 3.4.10 & Thm 3.4.11)

For (x_n) a bounded seq., limit superior of (x_n):

$$\limsup x_n \overset{\text{def}}{=} \inf\{v \in \mathbb{R} : v < x_n \text{ for finitely many } n\}$$
$$= \inf\{v \in \mathbb{R} : x_n \leq v \text{ for sufficiently large } n\}$$

And (i) If $v > \limsup x_n$, then

$x_n \leq v$ for sufficiently large n,

i.e. $\exists\, K(v) \in \mathbb{N}$ s.t. if $n \geq K(v)$, then $x_n \leq v$.

(ii) If $w < \limsup x_n$, then $\exists$ infinitely many $n \in \mathbb{N}$

s.t. $w \leq x_n$.

Def 9.4.8 Let
- $\sum a_n x^n$ be a power series, and
- $\rho = \begin{cases} \limsup(|a_n|^{\frac{1}{n}}), & \text{if } (|a_n|^{\frac{1}{n}}) \text{ is a bdd seq.} \\ +\infty, & \text{otherwise} \end{cases}$

Then
- the radius of convergence of $\sum a_n x^n$ is defined by

$$R = \frac{1}{\rho} = \begin{cases} 0, & \text{if } \rho = +\infty \\ \dfrac{1}{\limsup |a_n|^{\frac{1}{n}}}, & \text{otherwise } \left(\begin{array}{l}\text{including } R = +\infty \\ \text{when } \limsup|a_n|^{\frac{1}{n}} = 0\end{array}\right) \end{cases}$$

- The interval of convergence is the open interval $(-R, R)$

Thm 9.4.9 (Cauchy-Hadamard Theorem)

If R is the radius of convergence of $\sum a_n x^n$, then

$$\sum a_n x^n \text{ is } \begin{cases} \bullet \ \text{absolutely convergent} & \text{if } |x|<R, \\ \bullet \ \text{divergent} & \text{if } |x|>R. \end{cases}$$

Remark: No conclusion for $|x|=R$:

(i) $\sum x^n$: $\quad \rho = \limsup |a_n|^{\frac{1}{n}} = \limsup 1 = 1$

$$\Rightarrow R = \frac{1}{\rho} = 1.$$

$$\begin{cases} x=1: & \sum x^n = 1+1+1+\cdots \text{ is divergent} \\ x=-1: & \sum x^n = -1+1-1+1-\cdots \text{ is divergent.} \end{cases}$$

(ii) $\sum \frac{1}{n} x^n$: $\quad \rho = \limsup |a_n|^{\frac{1}{n}} = \limsup \left(\frac{1}{n}\right)^{\frac{1}{n}} = 1 \qquad$ (Ex!)

$$\Rightarrow R = \frac{1}{\rho} = 1$$

$$\begin{cases} x=1: & \sum \frac{1}{n} x^n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \text{ is divergent} \\ x=-1: & \sum \frac{1}{n} x^n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots \text{ is convergent.} \end{cases}$$

(iii) $\sum \frac{1}{n^2} x^n$: $\quad \rho = \limsup |a_n|^{\frac{1}{n}} = \limsup \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = 1 \qquad$ (Ex!)

$$\Rightarrow R = \frac{1}{\rho} = 1$$

$$\begin{cases} x=1: & \sum \frac{1}{n^2} x^n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \text{ is convergent.} \\ x=-1: & \sum \frac{1}{n^2} x^n = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \text{ is convergent.} \end{cases}$$

Pf of Cauchy-Hadamard Thm:

- $R=0$ and $R=+\infty$ leave as exercises.

Assume $0<R<+\infty$.

Clearly $\sum a_n x^n$ converges for $x=0$.

Consider $0<|x|<R$,

then $\exists\ 0<c<1$ such that $|x|<cR \ (=\frac{c}{\rho})$

Therefore $\rho|x| = \limsup(|a_n|^{\frac{1}{n}}|x|) < c$.

$\Rightarrow \exists K\in\mathbb{N}$ such that

if $n\geq K$, then $|a_n|^{\frac{1}{n}}|x| \leq c$.

$$\Rightarrow \quad |a_n x^n| \leq c^n, \ \forall\, n\geq K$$

Since $0<c<1$, $\sum c^n$ is convergent.

By Comparison Test (Thm 3.7.7), $\sum |a_n x^n|$ is convergent

i.e. $\sum a_n x^n$ is absolutely convergent.

This proves the 1st part.

If $|x|>R=\frac{1}{\rho}$, then $\rho = \limsup |a_n|^{\frac{1}{n}} > \frac{1}{|x|}$.

$\Rightarrow |a_n|^{\frac{1}{n}} > \frac{1}{|x|}$ for infinitely many $n\in\mathbb{N}$

i.e. $|a_n x^n|>1$ for infinitely many $n\in\mathbb{N}$

and hence $a_n x^n \not\to 0$ $\therefore$ $\sum a_n x^n$ is divergent. ⩩

Remarks: (i) $\boxed{\text{If } \lim \left|\frac{a_n}{a_{n+1}}\right| \text{ exists, then radius of convergence} = \lim \left|\frac{a_n}{a_{n+1}}\right|}$.

(Notes: • it is the reciprocal of $\left|\frac{a_{n+1}}{a_n}\right|$ in ratio test. (Ex 9.4.5)

• $\left|\frac{a_n}{a_{n+1}}\right| \to \infty$ is included)

When exists, it is usually easier to calculate:

(a) $\sum x^n$: $a_n = 1$, $\forall n$. $\left|\frac{a_n}{a_{n+1}}\right| \equiv 1 \to 1$ as $n \to \infty$

$\therefore\ R = 1$

(b) $\sum \frac{1}{n} x^n$: $a_n = \frac{1}{n}$, $\forall n$. $\left|\frac{a_n}{a_{n+1}}\right| = \frac{\frac{1}{n}}{\frac{1}{n+1}} = \frac{n+1}{n} \to 1$ as $n \to \infty$

$\therefore\ R = 1$

(c) $\sum \frac{1}{n^2} x^n$: $a_n = \frac{1}{n^2}$, $\forall n$. $\left|\frac{a_n}{a_{n+1}}\right| = \frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}} = \left(\frac{n+1}{n}\right)^2 \to 1$ as $n \to \infty$

$\therefore\ R = 1$

(ii) If one can choose $0 < C < 1$ independent of $x \in (-R, R)$, then one get uniform convergence. (Ex!)

Thm 9.4.10: Let • R = radius of convergence of $\sum a_n x^n$

• $[a,b] \subset (-R,R)$ be a closed and bounded interval.

Then $\sum a_n x^n$ converges uniformly on $[a,b]$.

Remark

- $R = +\infty$ included, and hence we need the assumption that $[a,b]$ is bounded.
- $R=0$ is excluded as $(-0,0) = \emptyset$.

(although $\sum a_n x^n$ always converges for $x=0$.

Note that we only have

interval of convergence $(-R,R) \subset$ domain of convergence,

they may not equal.)