Cor 9.2.9 $\begin{cases} \bullet\ x_n \neq 0,\ \forall n = 1,2,3,\cdots \\ \bullet\ a = \lim_{n\to\infty} n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right) \text{ exists} \end{cases}$

Then $\begin{cases} \bullet\ a > 1 \Rightarrow \sum x_n \text{ is absolutely convergent} \\ \bullet\ a < 1 \Rightarrow \sum x_n \text{ is not absolutely convergent} \end{cases}$

Pf: Omitted

Egs 9.2.10

(a) (limit) Raabe's Test for $\sum \frac{1}{n^p}$:

$$a = \lim_{n\to\infty} n\left(1 - \left|\frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}}\right|\right) = \lim_{n\to\infty} n\left(1 - \frac{n^p}{(n+1)^p}\right)$$

$$= \lim_{n\to\infty} n\left(1 - \frac{1}{(1+\frac{1}{n})^p}\right) = \lim_{n\to\infty}\left[\frac{(1+\frac{1}{n})^p - 1}{\frac{1}{n}} \cdot \frac{1}{(1+\frac{1}{n})^p}\right] \quad \text{(check!)}$$

Clearly $\lim_{n\to\infty} \frac{(1+\frac{1}{n})^p - 1}{\frac{1}{n}} = \left.\frac{d}{dx}\right|_{x=1} x^p = p$ (Thm 8.3.13)

$\therefore\ a = p \cdot 1 = p$.

By Cor 9.2.9 to Raabe's Test (or just call it (limit) Raabe's Test),

$p > 1 \Rightarrow \sum \frac{1}{n^p}$ is (absolutely) convergent

$p < 1 \Rightarrow \sum \frac{1}{n^p}$ is not (absolutely) convergent

(hence divergent (as $\frac{1}{n^p} > 0, \forall n$))

However, result for $p = 1$ cannot be deduced from Raabe's Test.

(b) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

Easy to check:

$$\begin{cases} \bullet \ \left|\frac{x_{n+1}}{x_n}\right| = \frac{\frac{n+1}{(n+1)^2+1}}{\frac{n}{n^2+1}} = \frac{n+1}{n} \cdot \frac{n^2+1}{(n+1)^2+1} \to 1, \text{ and} \\ \bullet \ n\left(1-\left|\frac{x_{n+1}}{x_n}\right|\right) = n \cdot \left(1 - \frac{n+1}{n} \cdot \frac{n^2+1}{(n+1)^2+1}\right) \end{cases}$$

$$= \frac{n^2+n-1}{(n+1)^2+1} \longrightarrow 1 \quad \text{as } n \to \infty$$

$\therefore$ Both Cor 9.2.5 and Cor 9.2.2 cannot be applied.

But
$$\left|\frac{x_{n+1}}{x_n}\right| - 1 = \frac{n+1}{n} \frac{n^2+1}{(n+1)^2+1} - 1 = \frac{(n+1)(n^2+1) - n[(n+1)^2+1]}{n[(n+1)^2+1]}$$

$$= -\frac{n^2+n-1}{n[(n+1)^2+1]} = -\frac{1}{n} \cdot \frac{n^2+n-1}{n^2+2n+2} \geq -\frac{1}{n} \quad \text{(check!)}$$

$$\therefore \left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{1}{n}, \ \forall n \geq 1 \quad \left(\begin{matrix} a = 1 \leq 1 \\ \& \ K = 1 \in \mathbb{N} \end{matrix}\right)$$

Raabe's Test (Thm 9.2.8) $\Rightarrow \sum x_n$ is not absolutely convergent.

Remarks: (i) "Limiting form" of Raabe's Test (Cor 9.2.9) doesn't apply, but Raabe's Test (Thm 9.2.8) applies.

(ii) Integral Test or Limit Comparison Test work for this example. (Ex!)

§ 9.3 Tests for Nonabsolute Convergence

Def 9.3.1 • $x_n \neq 0$, $\forall n \in \mathbb{N}$

Then • the sequence (x_n) is said to be alternating if $(-1)^{n+1} x_n > 0$ (or < 0) $\forall n \in \mathbb{N}$

• in this case, the series $\sum x_n$ is called an alternating series.

e.g. If $z_n > 0$, then $x_n = (-1)^{n+1} z_n$ and $x_n = (-1)^n z_n$ are alternating

(explicit eg: $z_n = \frac{1}{n} > 0$, $(x_n) = \left((-1)^{n+1} z_n\right) = (1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots)$ is alternating)

Thm 9.3.2 Let • $z_n > 0$ and decreasing ($z_{n+1} \leq z_n$) $\forall n \in \mathbb{N}$

• $\lim_{n \to \infty} z_n = 0$

Then the alternating series $\sum (-1)^{n+1} z_n$ is convergent

egs: By Thm 9.3.1, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ is convergent

(Note: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$ is divergent by integral test)

eg 9.2.7 (d)