Ch 9 Infinite Series

§9.1 Absolute Convergence

Recall Eg 3.7.6 (b) Harmonic series

$$\sum_{i=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \quad \text{is } \underline{\text{divergent}}$$

(since partial sum $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is unbounded)

but Eg 3.7.6 (f) Alternating harmonic series

$$\sum_{i=1}^{n} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \quad \text{is } \underline{\text{convergent}}$$

$\therefore$ A series $\sum x_n$ may be convergent, but the series $\sum |x_n|$ may be divergent

Def 9.1.1
- $\sum x_n$ is absolutely convergent if the series $\sum |x_n|$ is convergent
- $\sum x_n$ is conditionally convergent (or non-absolutely convergent) if $\sum x_n$ is convergent but $\sum |x_n|$ is divergent.

(i.e. conditionally convergent means convergent but not absolutely convergent)

Eg: Alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Thm 9.1.2 "Absolutely convergent" $\Rightarrow$ "convergent".

Pf: $\sum |x_n|$ convergent

$\Rightarrow \forall \varepsilon > 0, \exists M(\varepsilon) \in \mathbb{N}$ s.t. (Cauchy Criterion 3.7.4)

if $m > n \geq M(\varepsilon)$, then $|x_{n+1}| + \cdots + |x_m| < \varepsilon$

Let $S_n = x_1 + \cdots + x_n$ be the n^{th} partial sum of $\sum x_n$,

then $\forall\ m > n \geq M(\varepsilon)$,

$$|S_m - S_n| = |x_{n+1} + \cdots + x_m| \leq |x_{n+1}| + \cdots + |x_m| < \varepsilon.$$

$\therefore$ $\sum x_n$ is convergent. ※

<u>Grouping of Series</u>

For a series of $\sum x_n$, one can construct many other series $\sum y_k$ by "grouping the terms": inserting parentheses that group together finitely many terms, but keeping the order of the terms x_n fixed. That is

$$y_1 = \sum_{j=1}^{n_1} x_j, \quad y_2 = \sum_{j=n_1+1}^{n_2} x_j, \cdots, \quad y_k = \sum_{j=n_{k-1}+1}^{n_k} x_j, \cdots$$

$$(n_k < n_{k+1} \quad \forall\ k = 1, 2, \cdots \quad \& \quad n_0 = 0)$$

$\therefore$ $x_1 + x_2 + \cdots + x_n + \cdots$

$$= (x_1 + \cdots + x_{n_1}) + (x_{n_1+1} + \cdots + x_{n_2}) + (x_{n_2+1} + \cdots) + \cdots$$

$$= \quad y_1 \quad + \quad y_2 \quad + \quad y_3 \quad + \cdots$$

Eg: $1 - \frac{1}{2} + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6} + \frac{1}{7}\right) - \frac{1}{8} + \left(\frac{1}{9} - \cdots + \frac{1}{13}\right) - \cdots$

is a grouping the terms of the alternating harmonic series.

(i.e. $y_1 = 1$, $y_2 = -\frac{1}{2}$, $y_3 = \frac{1}{3} - \frac{1}{4}$, $y_4 = \frac{1}{5} - \frac{1}{6} + \frac{1}{7}$

$y_5 = -\frac{1}{8}$, $y_6 = \frac{1}{9} - \cdots + \frac{1}{13}$, $\ldots$)

Thm 9.1.3 $\sum x_n$ convergent

$\Rightarrow$ any series $\sum y_k$ obtained from it by grouping the terms is also convergent, & converges to the <u>same value</u>.

Pf: Let $s_n = n^{th}$ partial sum of $\sum x_n$

$t_k = k^{th}$ partial sum of $\sum y_k$.

If $y_k = \sum_{j=n_{k-1}+1}^{n_k} x_j$,

then $t_1 = y_1 = x_1 + \cdots + x_{n_1} = s_{n_1}$

$$t_2 = y_1 + y_2 = \sum_{j=1}^{n_1} x_j + \sum_{j=n_1+1}^{n_2} x_j = x_1 + \cdots + x_{n_2} = s_{n_2}$$

$\vdots$

$t_k = s_{n_k}$.

$\therefore$ (t_k) is a subseq. of (s_n)

Since $\sum x_n$ is convergent, $s_n \to s \left(= \sum_{n=1}^{\infty} x_n\right)$ as $n \to \infty$

$\therefore$ $t_k \to s$ as $k \to \infty$

i.e. $\sum y_k$ is convergent and converges to the same value as $\sum x_n$ $\#$

Remark: The converse of Thm 9.1.3 is not true.

Counterexample: Let $\sum x_n = 1-1+1-1+1 \cdots$

& $\sum y_k = (1-1)+(1-1)+(1-1)+\cdots$

Then $y_k = 0 \ \forall k \Rightarrow \sum y_k$ is convergent.

But original series $\sum x_n = 1-1+1-1+1 \cdots$ is divergent.

Rearrangement of series

(Not grouping any terms, but scrambling the order of the terms.)

> **Def 9.1.4** $\sum y_k$ is a rearrangement of $\sum x_n$,
>
> if $\exists$ a bijection (ie. one-to-one) $f : \mathbb{N} \to \mathbb{N}$ s.t.
>
> $$y_k = x_{f(k)} \quad \forall k \in \mathbb{N} = \{1,2,3,\cdots\}.$$

Remarks: (i) $\sum x_n$ is convergent $\not\Rightarrow$ $\sum y_k$ rearrangement is convergent

(Ex 9.1.3)

(ii) Riemann Thm: If $\sum x_n$ conditionally convergent,

then $\forall c \in \mathbb{R}$, $\exists$ a rearrangement $\sum y_k$ of $\sum x_n$ such that

$$\sum_{k=1}^{\infty} y_k = c \qquad \text{(Pf omitted)}$$

> **Thm 9.1.5** If $\sum x_n$ is absolutely convergent, then any rearrangement $\sum y_k$ of $\sum x_n$ converges to the same value.

Pf: $\sum x_n$ absolutely convergent $\Rightarrow$ $\sum x_n$ convergent.

Let $x = \sum_{n=1}^{\infty} x_n$, and $S_n = \sum_{k=1}^{n} x_k$.

Then $S_n \to x$ as $n \to \infty$

$\therefore$ $\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t.

$$\text{if } n \geq N_1, \quad |S_n - x| < \varepsilon.$$

On the other hand, $\sum |x_n|$ convergent

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists N_2 \in \mathbb{N}$ s.t.

if $q > l \geq N_2$, then $|x_{l+1}| + |x_{l+2}| + \cdots + |x_q| < \varepsilon$

Therefore, for $N = \max\{N_1, N_2\}$,

if $n, q > N$,

$$\begin{cases} |S_n - x| < \varepsilon \text{ and} \\ |x_{N+1}| + |x_{N+2}| + \cdots + |x_q| < \varepsilon \end{cases} \qquad \text{—} (*)$$

Let $\sum y_k$ be a rearrangement of $\sum x_n$ given by the bijection $f: \mathbb{N} \to \mathbb{N}$, i.e. $y_k = x_{f(k)}$, $\forall k \in \mathbb{N}$.

Let $M = \max\{f^{-1}(1), \cdots, f^{-1}(N)\}$,

then all the terms $x_1, \cdots, x_N$ are contained in $\{y_1, \cdots, y_M\}$.

$\therefore$ If $t_m = \sum_{k=1}^{m} y_k$, then $\forall m \geq M$, $(\& \ n > N)$

$$t_m - S_n = (y_1 + \cdots + y_M + \cdots + y_m) - (x_1 + \cdots + x_N + \cdots + x_n)$$

$$= \underbrace{(y_1+\cdots+y_M)-(x_1+\cdots+x_N)}_{(\text{no } x_1,\cdots,x_N \text{ remain})} + \underbrace{(y_{M+1}+\cdots+y_m)-(x_{N+1}+\cdots+x_n)}_{(\text{no } x_1,\cdots,x_N \text{ in these terms})}$$

is a sum of finite number of terms x_k with $k>N$.

$$\Rightarrow \quad |t_m - S_n| \le \sum_{k=N+1}^{q} |x_k| \quad \text{for some } q$$

By $(*)$, $|t_m - S_n| < \varepsilon$.

Hence, $\forall \varepsilon>0$, $\exists M>0$ such that

if $m \ge M$, $|t_m - x| \le |t_m - S_n| + |S_n - x| < \varepsilon + \varepsilon = 2\varepsilon$.

(using a term with $n>N$)

Since $\varepsilon>0$ is arbitrary, $\lim_{m\to\infty} t_m = x$

$\therefore \sum y_k \longrightarrow x = \sum x_n$.

§9.2 Tests for Absolute Convergence

Thm 9.2.1 (Limit Comparison Test II)

Suppose $\begin{cases} \bullet\ x_n, y_n \neq 0, \ \forall n = 1, 2, \cdots \\ \bullet\ \lim_{n\to\infty} \left| \frac{x_n}{y_n} \right| = r \text{ exists} \end{cases}$

Then (a) If $r \neq 0$, then

$$\sum x_n \text{ absolutely convergent} \Leftrightarrow \sum y_n \text{ absolutely convergent}$$

(b) If $r = 0$ and $\sum y_n$ absolutely convergent,

then $\sum x_n$ is absolutely convergent (only $\sum y_n \Rightarrow \sum x_n$, $\not\Leftarrow$ in this case)

Pf: Recall Limit Comparison Test (Thm 3.7.8) that

if $x_n, y_n > 0$, $r = \lim_{n\to\infty} \frac{x_n}{y_n}$ exists

then $\begin{cases} \bullet\ \text{If } r \neq 0,\ \sum x_n \text{ converges} \Leftrightarrow \sum y_n \text{ converges} \\ \bullet\ \text{If } r = 0,\ \sum y_n \text{ converges} \Rightarrow \sum x_n \text{ converges.} \end{cases}$

Applying Thm 3.7.8 to $\sum |x_n|$ & $\sum |y_n|$ ※

Recall also Comparison Test (Thm 3.7.7): $0 \le x_n < y_n$, $\forall n \ge K$ (for some $K \in \mathbb{N}$)

then $\begin{cases} \text{(a) } \sum y_n \text{ converges} \Rightarrow \sum x_n \text{ converges} \\ \text{(b) } \sum x_n \text{ diverges} \Rightarrow \sum y_n \text{ diverges.} \end{cases}$

Thm 9.2.2 (**Root Test**) (Cauchy)

(a) If $\exists\, \underline{r<1}$ and $K \in \mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} \le r, \quad \forall\, n \ge K,$$

then $\sum x_n$ is <u>absolutely</u> convergent.

(b) If $\exists\, K \in \mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} \ge 1, \quad \forall\, n \ge K,$$

then $\sum x_n$ is <u>divergent</u>.

<u>Pf</u>: (a) If $|x_n|^{\frac{1}{n}} \le r$, $\forall\, n \ge K$

then $|x_n| \le r^n$, $\forall\, n \ge K$

Since $\sum r^n$ is convergent for $0 \le r < 1$,

<u>Comparison Test 3.7.7</u> $\Rightarrow$ $\sum |x_n|$ is convergent.

(b) If $|x_n|^{\frac{1}{n}} \ge 1$, then $|x_n| \ge 1$, $\forall\, n \ge K$

$\Rightarrow$ $x_n \not\to 0$ as $n \to \infty$

$\Rightarrow$ $\sum x_n$ is divergent (n^{th} Term Test 3.7.3) ⋕

Cor 9.2.3 Suppose $r = \lim\limits_{n\to\infty} |x_n|^{\frac{1}{n}}$ exists.

Then $\begin{cases} \bullet\ \underline{r<1} \Rightarrow \sum x_n \text{ is } \underline{\text{absolutely convergent}} \\ \bullet\ \underline{r>1} \Rightarrow \sum x_n \text{ is } \underline{\text{divergent}}. \end{cases}$

(No conclusion for $r=1$. see Eg 9.2.7(b) later)

Pf: If $r<1$, then $\forall\ r<r_1<1$, $\exists\ K\in\mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} \le r_1 < 1, \quad \forall\ n \ge K,$$

then part (a) of Root Test $\Rightarrow$ $\sum x_n$ absolutely convergent.

If $r>1$, then $\exists\ K\in\mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} > 1, \quad \forall\ n \ge K,$$

then part (b) of Root Test $\Rightarrow$ $\sum x_n$ divergent. #

Thm 9.2.4 (Ratio Test) (D'Alembert)

Let $x_n \ne 0$, $\forall\ n=1,2,3,\cdots$

(a) If $\exists\ 0<r<1$ and $K\in\mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \le r, \quad \forall\ n \ge K,$$

then $\sum x_n$ is absolutely convergent

(b) If $\exists\ K\in\mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \ge 1, \quad \forall\ n \ge K,$$

then $\sum x_n$ is divergent.

Pf: (a) $\forall\ n\ge K$, $|x_n| \le r|x_{n-1}| \le r^2|x_{n-2}| \le \cdots \le r^{n-K}|x_K|$

If $0<r<1$, then $\sum y_n \overset{def}{=} \sum r^{n-K}|x_K| = \frac{|x_K|}{r^K}\sum r^n$ is convergent

Comparison Test 3.7.7 $\Rightarrow$ $\sum |x_n|$ is convergent.

i.e. $\sum x_n$ is absolutely convergent.

(b) $\forall n \geq K$, $|x_n| \geq |x_{n-1}| \geq |x_{n-2}| \geq \cdots \geq |x_K|$

$\therefore$ $x_n \not\to 0$ as $n \to \infty$ $\Rightarrow$ $\sum x_n$ is divergent. ✱

<u>Cor 9.2.5</u> If $\begin{cases} \bullet\ x_n \neq 0,\ \forall n = 1, 2, 3, \cdots, \text{ and} \\ \bullet\ r = \lim\limits_{n\to\infty} \left| \dfrac{x_{n+1}}{x_n} \right| \text{ exists} \end{cases}$

Then $\begin{cases} \bullet\ r < 1 \Rightarrow \sum x_n \text{ is } \underline{\text{absolutely convergent}}. \\ \bullet\ r > 1 \Rightarrow \sum x_n \text{ is } \underline{\text{divergent}} \end{cases}$

(No conclusion for $r = 1$. see Eg 9.2.7(c) later)

<u>Pf</u>: If $r < 1$, then $\forall r_1 \in (r, 1)$, $\exists K \in \mathbb{N}$ s.t.

$$\left| \frac{x_{n+1}}{x_n} \right| < r_1 < 1, \quad \forall n \geq K$$

Part (a) of Thm 9.2.4 $\Rightarrow$ $\sum x_n$ is absolutely convergent.

If $r > 1$, then $\exists K \in \mathbb{N}$ s.t.

$$\left| \frac{x_{n+1}}{x_n} \right| > 1, \quad \forall n \geq K$$

Part (b) of Thm 9.2.4 $\Rightarrow$ $\sum x_n$ is divergent. ✱

The Integral Test

Def (Improper Integral)

For $a \in \mathbb{R}$, if $\begin{cases} \bullet\ f \in R[a,b],\ \forall b > a, \text{ and} \\ \bullet\ \lim_{b\to+\infty} \int_a^b f \text{ exists (and } < +\infty.) \end{cases}$

then the improper integral $\int_a^\infty f$ is defined to be

$$\int_a^\infty f = \lim_{b\to+\infty} \int_a^b f .$$

Thm 9.2.6 (Integral Test)

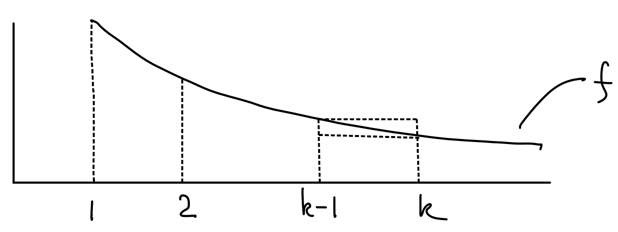
Let $f(t) > 0$, decreasing on $\{t \geq 1\}$.

Then $\sum_{k=1}^{\infty} f(k)$ converges $\Leftrightarrow$ $\int_1^\infty f = \lim_{b\to+\infty} \int_1^b f$ exists.

In this case,

$$\int_{n+1}^{\infty} f(t)\,dt \leq \sum_{k=1}^{\infty} f(k) - \sum_{k=1}^{n} f(k) \leq \int_n^\infty f(t)\,dt, \quad \forall n = 1, 2, \cdots$$

Pf:



$f > 0$ & decreasing $\Rightarrow$ $\forall k = 2, 3, \cdots$

$$f(k) \leq \int_{k-1}^{k} f(t)\,dt \leq f(k-1) \qquad — \quad (*)_1$$

$$\Rightarrow \sum_{k=2}^{n} f(k) \leq \sum_{k=2}^{n} \int_{k-1}^{k} f(t)\,dt \leq \sum_{k=2}^{n} f(k-1) \quad \Big(= f(1) + \cdots + f(n-1)\Big)$$

Let $S_n = \sum_{k=1}^{n} f(k)$

Then, we have

$$S_n - f(1) \le \int_1^n f(t)\,dt \le S_{n-1} .$$

$\therefore \lim_{n\to\infty} S_n$ exists $\Leftrightarrow \lim_{n\to\infty} \int_1^n f(t)\,dt$ exist (bdd, increasing)

& $\sum_{k=1}^{\infty} f(k)$ converges $\Leftrightarrow \int_1^{\infty} f$ exists. ($\lim_{n\to\infty} \int_1^n f$ exists $\Rightarrow \lim_{b\to\infty} \int_1^b f$ exists)

see below

Using $(*)_1$ again, if $m > n$, then

$$\sum_{k=n+1}^{m} f(k) \le \sum_{k=n+1}^{m} \int_{k-1}^{k} f(t)\,dt \le \sum_{k=n+1}^{m} f(k-1)$$

$$\Rightarrow \quad S_m - S_n \le \int_n^m f(t)\,dt \le S_{m-1} - S_{n-1}$$

Hence, $\forall\, m > n$, we have

$$\int_{n+1}^{m+1} f(t)\,dt \le S_m - S_n \le \int_n^m f(t)\,dt$$

Letting $m \to \infty$, we have

$$\int_{n+1}^{\infty} f(t)\,dt \le S - S_n \le \int_n^{\infty} f(t)\,dt$$

where $S = \sum_{k=1}^{\infty} f(k)$.

Finally,

<u>Claim</u>: Let $f > 0$, on $[1,\infty)$; $f \in R[1,b]$, $\forall\, b > 1$, then

$$\lim_{n\to\infty} \int_1^n f(t)\,dt \text{ exists} \Leftrightarrow \lim_{b\to\infty} \int_1^b f(t)\,dt \text{ exists}.$$

Pf: ($\Rightarrow$) Assume $\lim_{n\to\infty} \int_1^n f$ exists.

$\forall b > 1$, $\exists n \in \mathbb{N}$ s.t. $n \leq b < n+1$

(in fact n = largest integer $\leq b$.)

Since $f > 0$,
$$\int_1^n f(t)dt \leq \int_1^b f(t)\,dt \leq \int_1^{n+1} f(t)\,dt$$

※

Since $b \to \infty \Rightarrow n \to \infty$, and
$$\lim_{n\to\infty} \int_1^n f(t)dt = \lim_{n\to\infty} \int_1^{n+1} f(t)dt$$

$\therefore$ $\lim_{b\to\infty} \int_1^b f(t)\,dt$ exists and $= \lim_{n\to\infty} \int_1^n f(t)dt$

($\Leftarrow$) Assume $\lim_{b\to\infty} \int_1^b f$ exists.

Then subseq $\int_1^n f$ has limit & equals $\lim_{b\to\infty} \int_1^b f$.

This completes the proof of the integral test. ※

Egs 9.2.7

(a) Recall Eg 3.7.2(c)

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$ is (absolutely) convergent. $\left(\frac{1}{n(n+1)} > 0\right)$

Using Limit Comparison Test II (Thm 9.2.1)

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n(n+1)}} = \lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$$

$\Rightarrow$ $\sum \frac{1}{n^2}$ is (absolutely) convergent.

(b) However, Root Test (Thm 9.2.2) doesn't apply to $\sum \frac{1}{n^2}$ (in fact $\sum \frac{1}{n^p}$, $\forall p > 0$):

- $\left(\frac{1}{n^p}\right)^{\frac{1}{n}} < 1$, and
- $\left|\frac{1}{n^p}\right|^{1/n} = \frac{1}{(n^{\frac{1}{n}})^p} \longrightarrow 1$ since $n^{\frac{1}{n}} \to 1$

$\therefore$ both conditions in part (a) & part (b) don't hold.

And the Cor 9.2.3 cannot be applied too.

$$\left(r = \lim_{n\to\infty} \left|\frac{1}{n^p}\right|^{\frac{1}{n}} = 1.\right)$$

(c) Ratio Test (Thm 9.2.4) and its Cor 9.25 also don't work for $\sum \frac{1}{n^p}$:

$$\left| \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} \right| = \frac{n^p}{(n+1)^p} = \frac{1}{(1+\frac{1}{n})^p} \to 1$$

$r = 1$, no information from Ratio test!

(d) On the other hand, Integral Test (Thm 9.2.6) works for $\sum \frac{1}{n^p}$:

Let $f(t) = \frac{1}{t^p}$, $t \geq 1$.

Then $f(t) > 0$ and decreasing.

$$\lim_{n\to\infty} \int_1^n \frac{1}{t^p}\,dt = \begin{cases} \lim_{n\to\infty} \left(\log(n) - \log 1\right), & p = 1 \\ \lim_{n\to\infty} \left[\frac{t^{1-p}}{1-p}\right]_1^n, & p \neq 1 \end{cases}$$

(same as $b \to \infty$)

Since $\log(n) \to +\infty$, $\int_1^\infty \frac{1}{t}\,dt$ doesn't exist

$$\lim_{n\to\infty} \frac{1}{1-p}\left(\frac{1}{n^{p-1}} - 1\right) = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ +\infty, & \text{if } p < 1 \end{cases}$$

$\therefore \int_1^\infty \frac{1}{t^p}\,dt$ $\begin{cases} \text{exists if } p > 1 \\ \text{doesn't exist if } p < 1. \end{cases}$

Altogether, $\sum \frac{1}{n^p}$ $\begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1. \end{cases}$ ∗

Thm 9.2.8 (**Raabe's Test**) Suppose $x_n \neq 0$, $\forall n = 1, 2, 3, \cdots$

(a) If $\exists$ $\underline{a > 1}$ and $K \in \mathbb{N}$ s.t.

$$\left| \frac{x_{n+1}}{x_n} \right| \leq 1 - \frac{a}{n}, \quad \forall n \geq K$$

(Note: This condition allows $\lim \left| \frac{x_{n+1}}{x_n} \right| = 1$)

then $\sum x_n$ is <u>absolutely convergent</u>

(b) If $\exists$ $\underline{a \leq 1}$ and $K \in \mathbb{N}$ s.t.

$$\left| \frac{x_{n+1}}{x_n} \right| \geq 1 - \frac{a}{n} \quad \forall n \geq K$$

(Note: This condition allows $\lim \left| \frac{x_{n+1}}{x_n} \right| = 1$)

then $\sum x_n$ is <u>not</u> <u>absolutely</u> convergent.

<u>Pf</u>: Omitted