§8.4 The Trigonometric Functions

Thm 8.4.1 $\exists$ functions $C:\mathbb{R}\to\mathbb{R}$ and $S:\mathbb{R}\to\mathbb{R}$ such that

(i) $C''(x) = -C(x)$ and $S''(x) = -S(x)$, $\forall x\in\mathbb{R}$.

(ii) $\begin{cases} C(0)=1 \\ C'(0)=0 \end{cases}$ and $\begin{cases} S(0)=0 \\ S'(0)=1 \end{cases}$

Pf: Define $C_n(x)$ and $S_n(x)$ inductively by

$$\begin{cases} C_1(x)=1 \\ S_1(x)=x \end{cases}$$

$$\begin{cases} S_n(x)=\int_0^x C_n(t)\,dt \\ C_{n+1}(x)=1-\int_0^x S_n(t)\,dt \end{cases}$$

(i.e. starting with C_1: $C_1 \to S_1 \nearrow C_2 \to S_2 \nearrow C_3 \to S_3 \nearrow \cdots\ C_n \to S_n \nearrow C_{n+1} \to \cdots$)

Then "Induction": C_n & S_n are continuous, $\forall n$

$\Rightarrow$ integrable on any bounded interval

$\therefore$ All C_n & S_n are well-defined.

Moreover, by Fundamental Thm 7.3.5,

$$S_n'(x) = C_n(x) \quad \& \quad C_{n+1}'(x) = -S_n(x), \quad \forall x\in\mathbb{R}, \forall n$$

Claim:

$$\begin{cases} C_{n+1}(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots + (-1)^n \dfrac{x^{2n}}{(2n)!} \\[2ex] S_{n+1}(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} \end{cases}$$

Pf: (Ex! By induction)

Now let $A > 0$.

If $x \in [-A, A]$ (i.e. $|x| \le A$) and $m > n > 2A$, $\left(\frac{A}{2n}, \frac{A}{2m} < \frac{1}{4} \right)$

then

$$|C_m(x) - C_n(x)| = \left| (-1)^n \frac{x^{2n}}{(2n)!} + \cdots + (-1)^{m-1} \frac{x^{2(m-1)}}{(2(m-1))!} \right|$$

$$\le \frac{A^{2n}}{(2n)!} + \cdots + \frac{A^{2m-2}}{(2m-2)!}$$

$$= \frac{A^{2n}}{(2n)!} \left[1 + \frac{(2n)!}{(2(n+1))!} A^2 + \frac{(2n)!}{(2(n+2))!} A^4 + \cdots + \frac{(2n)!}{(2(m-1))!} A^{2(m-1-n)} \right]$$

$$\le \frac{A^{2n}}{(2n)!} \left[1 + \frac{A^2}{(2n)^2} + \frac{A^4}{(2n)^4} + \cdots + \frac{A^{2(m-1-n)}}{(2n)^{2(m-1-n)}} \right]$$

$$\le \frac{A^{2n}}{(2n)!} \left[1 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^4 + \cdots + \left(\frac{1}{4}\right)^{2(m-1-n)} \right]$$

$$< \frac{16}{15} \cdot \frac{A^{2n}}{(2n)!}$$

Since $\lim_{n \to \infty} \frac{A^{2n}}{(2n)!} = 0$, Cauchy Criterion for Uniform Convergence implies C_n converges uniformly on $[-A, A]$, $\forall A > 0$

And hence, $C_n(x)$ converges $\forall x \in \mathbb{R}$.

Let $C(x) = \lim_{n\to\infty} C_n(x)$.

Then C_n converges uniformly to C on $[-A, A]$, $\forall A > 0$.

Hence Thm 8.2.2 $\Rightarrow$

C is cts on $[-A, A]$, $\forall A > 0$

and therefore, C is cts on $\mathbb{R}$.

Moreover, $C_n(0) = 1$, $\forall n \Rightarrow C(0) = 1$.

Since $S_n(x) = \int_0^x C_n(t)\, dt$

$$S_m(x) - S_n(x) = \int_0^x (C_m(t) - C_n(t))\, dt$$

$\Rightarrow$ $|S_m(x) - S_n(x)| \le \int_0^x |C_m(t) - C_n(t)|\, dt$ if $x \ge 0$

(Cor 7.3.15) $\left(\int_x^0 |C_m(t) - C_n(t)|\, dt, \text{ if } x < 0\right)$

Then for $x \in [-A, A]$ & $m > n > 2A$,

$$|S_m(x) - S_n(x)| \le \int_0^x \frac{16}{15} \cdot \frac{A^{2n}}{(2n)!}\, dt$$

$$\le \frac{16}{15} \cdot \frac{A^{2n}}{(2n)!} \cdot A \quad \left(\text{similarly for } \int_x^0 \cdots\right)$$

$$\to 0 \text{ as } n \to \infty$$

$\therefore$ S_n converges uniformly on $[-A, A]$, $\forall A > 0$.

$\Rightarrow$ $S_n(x)$ converges $\forall x \in \mathbb{R}$.

Let $S(x) = \lim_{n\to\infty} S_n(x)$, $\forall x \in \mathbb{R}$

Then S_n converges uniformly to S on $[-A, A]$, $\forall A > 0$.

By Thm 8.2.2, S is cts on $\mathbb{R}$ (as S_n cts on $\mathbb{R}$, $\forall n$)

Since $S_n(0) = 0$, $\forall n$, we have $S(0) = 0$.

Now by Fundamental Thm of Calculus,

$$C_n'(x) = -S_{n-1}(x) \underset{\text{(uniform)}}{\rightrightarrows} -S(x) \quad \text{on } [-A, A], \ \forall A > 0$$

Thm 8.2.3 $\Rightarrow$

$$C(x) = \lim_{n\to\infty} C_n(x) \quad \text{is differentiable and}$$

$$C'(x) = -S(x) \quad \text{on } [-A, A], \ \forall A > 0$$

Hence C is differentiable $\forall x \in \mathbb{R}$ and

$$C'(x) = -S(x), \ \forall x \in \mathbb{R}.$$

In particular, $C'(0) = -S(0) = 0$

Similarly, Fundamental Thm

$$\Rightarrow S_n'(x) = C_n(x) \rightrightarrows C(x) \quad \text{on } [-A, A], \ \forall A > 0$$

$\vdots$ (Ex!)

$\Rightarrow$ S is differentiable $\forall x \in \mathbb{R}$ &

$$S'(x) = C(x), \ \forall x \in \mathbb{R}$$

In particular, $S'(0) = C(0) = 1$.

Finally, combining the 2 formulae of 1^{st} derivatives, we have

$$C''(x) = -S'(x) = -C(x) \quad \&$$

$$S''(x) = C'(x) = -S(x) .$$

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Cor 8.4.2 If C, S are the functions in Thm 8.4.1, then

$$\text{(iii)} \begin{cases} C'(x) = -S(x), \\ S'(x) = C(x) \end{cases} \quad \forall x \in \mathbb{R}.$$

Moreover, C & S have derivatives of all orders

Pf: Easy

Cor 8.4.3 The functions C & S in Thm 8.4.1 satisfy the Pythagorean Identity: $(C(x))^2 + (S(x))^2 = 1, \quad \forall x \in \mathbb{R}$

Pf: Let $f(x) = (C(x))^2 + (S(x))^2$.

By Thm 8.4.1, f is differentiable &

$$f'(x) = 2C(x)C'(x) + 2S(x)S'(x)$$

$$= -2C(x)S(x) + 2S(x)C(x) = 0, \quad \forall x \in \mathbb{R}$$

$\Rightarrow$ $f(x)$ is a constant function on $\mathbb{R}$.

$\Rightarrow$ $f(x) \equiv f(0) = (C(0))^2 + (S(0))^2 = 1, \quad \forall x \in \mathbb{R}.$ ※

Thm 8.4.4 The functions C and S satisfying

$$(*)_C \begin{cases} C'' = -C \\ C(0) = 1 \\ C'(1) = 0 \end{cases} \quad \text{and} \quad (*)_S \begin{cases} S'' = -S \\ S(0) = 0 \\ S'(0) = 1 \end{cases}$$

are unique.

Pf: Omitted (similar argument as in the proof for exponential function E by using Taylor's Thm, but reduce to "two" terms instead of "one" because the equations are 2nd order.)

Def 8.4.5 The unique functions C & S given in Thm 8.4.1 are called the cosine function and the sine function respectively, and denoted by

$$\cos x = C(x) \quad \& \quad \sin x = S(x)$$

Thm 8.4.6: If $f: \mathbb{R} \to \mathbb{R}$ satisfies $f''(x) = -f(x)$, $\forall x \in \mathbb{R}$, then $\exists$ real numbers α, β such that

$$f(x) = \alpha C(x) + \beta S(x), \quad \forall x \in \mathbb{R}.$$

Pf: Let $\alpha = f(0)$ & $\beta = f'(0)$.

and consider $h(x) = f(x) - [\alpha C(x) + \beta S(x)]$, $\forall x \in \mathbb{R}$.

Then it is easy to check that (Ex!)

$$\begin{cases} h'' = -h \\ h(0) = 0 \\ h'(0) = 0 \end{cases}$$

Similarly argument as in the proof of Thm 8.4.4,

we have $h(x) \equiv 0, \ \forall x \in \mathbb{R}$.

$\therefore \quad f(x) = \alpha C(x) + \beta S(x) \quad \forall x \in \mathbb{R}$.

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Thm 8.4.7 The cosine $C(x)$ & sine $S(x)$ satisfy

(v) $C(-x) = C(x)$ & $S(-x) = -S(x) \quad \forall x \in \mathbb{R}$

(vi) $\begin{cases} C(x+y) = C(x)C(y) - S(x)S(y) \\ S(x+y) = S(x)C(y) + C(x)S(y) \end{cases}$ (compound angle formulae)

Pf: Omitted (Easy by Thm 8.4.4 & 8.4.6)

Thm 8.4.8 For $x \geq 0$,

(vii) $-x \leq S(x) \leq x$;

(viii) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1$;

(ix) $x - \frac{1}{6}x^3 \leq S(x) \leq x$;

(x) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

Pf: Omitted

Lemma 8.4.9
- $\exists$ a root γ of $C(x)$ in the interval $[\sqrt{2}, \sqrt{3})$.
- Moreover, $C(x) > 0 \ \forall x \in [0, \gamma)$.
- The number 2γ is the smallest positive root of $S(x)$.

Pf: (Ex!)

(Using (X) in Thm 8.4.8, continuity of $C(x)$ & (vi) in Thm 8.4.7)

Note: Of course, we can prove that $\gamma > \sqrt{2}$ as stated in the Textbook. But we need Ex 8.4.4 (not just Thm 8.4.8).

Def 8.4.10 $\pi \overset{\text{def}}{=} 2\gamma$ = smallest positive root of S

Note: Thm 8.4.8 (X) $\Rightarrow$ $2.828 \leq \pi < 2 \times \underbrace{\sqrt{6-2\sqrt{3}}}_{\text{smallest positive root of } 1-\frac{1}{2}x^2+\frac{1}{24}x^4} < 3.185$ (Ex!)

Thm 8.4.11
- C & S are 2π-periodic (have period 2π)

(xi) $C(x+2\pi) = C(x)$ & $S(x+2\pi) = S(x)$, $\forall x \in \mathbb{R}$

- $\begin{cases} S(x) = C(\frac{\pi}{2}-x) = -C(x+\frac{\pi}{2}) \\ C(x) = S(\frac{\pi}{2}-x) = S(x+\frac{\pi}{2}) \end{cases} \quad \forall x \in \mathbb{R}$

Pf Omitted.