The Logarithm Function

Def 8.3.8 The inverse function of E is called the logarithm (or the natural logarithm).

Notation: In the Textbook, logarithm is denoted by "L".

Other common notations are "$\ln$" or "$\log$".

(more common in graduate textbook of research articles in mathematics)

Note: By definition

$$\begin{cases} (L \circ E)(x) = x, & \forall x \in \mathbb{R} \\ (E \circ L)(y) = y, & \forall y > 0 \end{cases} \qquad (E : \mathbb{R} \to \{y > 0\} = E(\mathbb{R}))$$

i.e. $\ln e^x = x$, $e^{\ln y} = y$

(or $\log e^x = x$, $e^{\log y} = y$)

Thm 8.3.9 • The logarithm $L:\{x>0\}\to\mathbb{R}$ is a <u>strictly increasing</u> function with domain $\{x\in\mathbb{R}: x>0\}$ and $L(\{x>0\})=\mathbb{R}$.

- $L'(x)=\frac{1}{x}, \ \forall x>0$ —— (vii)
- $L(xy)=L(x)+L(y), \ \forall x>0, y>0$ —— (viii)
- $L(1)=0$ & $L(e)=1$ —— (ix)
- $L(x^r)=rL(x), \ \forall x>0$ and $r\in\mathbb{Q}$ —— (x)
- $\lim_{x\to 0^+} L(x)=-\infty$ & $\lim_{x\to+\infty} L(x)=+\infty$ —— (xi)

Pf: All are easy from the definition & properties of $E(x)$. (Ex!)

[Note that in property (x), $L(x^r)=rL(x)$ actually works for irrational number α: $L(x^\alpha)=\alpha L(x)$.

Although we used it a lot, x^α is not yet defined in the Textbook for $\alpha\notin\mathbb{Q}$.]

Power Functions

Def 8.3.10 If $\alpha\in\mathbb{R}$ and $x>0$, then

$$x^\alpha \overset{\text{def}}{=\!=} e^{\alpha\ln x} = E(\alpha L(x)) = e^{\alpha\log x}$$

The function $x\mapsto x^\alpha$ for $x>0$ is called the <u>power function with exponent α</u>.

<u>Note</u>: If $\alpha = r \in \mathbb{Q}$, then for $x > 0$

$$E(\alpha L(x)) = E(rL(x)) = E(L(x^r)) \quad \text{(by property (x))}$$
$$= x^r$$

$\therefore$ Def 8.3.10 is consistent with previous definition for $r \in \mathbb{Q}$.

<u>Thm 8.3.11</u> If $\alpha \in \mathbb{R}$, $x, y \in (0, \infty)$, then

(a) $1^\alpha = 1$,

(b) $x^\alpha > 0$,

(c) $(xy)^\alpha = x^\alpha y^\alpha$,

(d) $\left(\frac{x}{y}\right)^\alpha = \frac{x^\alpha}{y^\alpha}$

Pf: (Easy Ex!)

<u>Thm 8.3.12</u> If $\alpha, \beta \in \mathbb{R}$, $x \in (0, \infty)$, then

(a) $x^{\alpha+\beta} = x^\alpha x^\beta$,

(b) $(x^\alpha)^\beta = x^{\alpha\beta} = (x^\beta)^\alpha$,

(c) $x^{-\alpha} = \frac{1}{x^\alpha}$,

(d) If $\alpha < \beta$, then $x^\alpha < x^\beta$ for $x > 1$

Pf: (Easy Ex!)

Thm 8.3.13 For $\alpha \in \mathbb{R}$,

$x \mapsto x^{\alpha}$ is continuous and differentiable on $(0, \infty)$, and

$$D x^{\alpha} = \alpha x^{\alpha - 1}$$

Pf: Chain rule $\Rightarrow$ x^{α} is differentiable & hence continuous

and
$$D x^{\alpha} = D\big(E(\alpha L(x))\big) = E'(\alpha L(x))\, D(\alpha L(x))$$
$$= E(\alpha L(x)) \cdot \alpha\, D(L(x))$$
$$= \alpha x^{\alpha} \cdot \frac{1}{x}$$
$$= \alpha x^{\alpha - 1}.$$

The Function $\log_a$ (logarithm of x to the base a)

Def 8.3.14 Let $a > 0$ and $a \neq 1$.

$$\log_a(x) \overset{\text{def}}{=\!=} \frac{\ln x}{\ln a} = \frac{\log x}{\log a} \quad \text{for } x > 0.$$